

# Approximate Unitary Equivalence in Simple $C^*$ -algebras of Tracial Rank One

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## Abstract

Let  $C$  be a unital AH-algebra and let  $A$  be a unital separable simple  $C^*$ -algebra with tracial rank no more than one. Suppose that  $\phi, \psi : C \rightarrow A$  are two unital monomorphisms. With some restriction on  $C$ , we show that  $\phi$  and  $\psi$  are approximately unitarily equivalent if and only if

$$\begin{aligned} [\phi] &= [\psi] \text{ in } KL(C, A) \\ \tau \circ \phi &= \tau \circ \psi \text{ for all tracial states of } A \text{ and} \\ \phi^\dagger &= \psi^\dagger, \end{aligned}$$

where  $\phi^\dagger$  and  $\psi^\dagger$  are homomorphisms from  $U(C)/CU(C) \rightarrow U(A)/CU(A)$  induced by  $\phi$  and  $\psi$ , respectively, and where  $CU(C)$  and  $CU(A)$  are *closures* of the subgroup generated by commutators of the unitary groups of  $C$  and  $B$ .

A more practical but approximate version of the above is also presented.

## 1 Introduction

Let  $T_1$  and  $T_2$  be two normal operators in  $M_n$ , the algebra of  $n \times n$  matrices. Then  $T_1$  and  $T_2$  are unitarily equivalent, or, there exists a unitary  $U$  such that  $U^*T_1U = T_2$  if and only if

$$sp(T_1) = sp(T_2)$$

counting the multiplicities. Let  $X = sp(T_1)$ . Define  $\phi_i : C(X) \rightarrow M_n$  by

$$\phi(f) = f(T_i) \text{ for } f \in C(X), \ i = 1, 2.$$

Let  $\tau : M_n \rightarrow \mathbb{C}$  be the normalized tracial state on  $M_n$ . Then  $\tau \circ \phi_i$  ( $i = 1, 2$ ) gives a Borel probability measure  $\mu_i$  on  $C(X)$ ,  $i = 1, 2$ . Then  $\phi_1$  and  $\phi_2$  are unitarily equivalent if and only if  $\mu_1 = \mu_2$ . More generally, one may formulate the following theorem:

**1.1.** *Let  $X$  be a compact metric space and let  $\phi_1, \phi_2 : C(X) \rightarrow M_n$  be two homomorphisms. Then  $\phi_1$  and  $\phi_2$  are unitarily equivalent if and only if*

$$\tau \circ \phi_1 = \tau \circ \phi_2. \tag{e1.1}$$

For infinite dimensional situation, one has the following classical result: two bounded normal operators on an infinite dimensional separable Hilbert space are unitarily equivalent if and only if they have the same equivalent spectral measures and multiplicity functions (cf. Theorem 10.21 of [5]). Perhaps a more interesting and useful statement is the following: Let  $T_1$  and  $T_2$  be two bounded normal operators in  $B(l^2)$ . Then there exists a sequence of unitary  $U_n \in B(l^2)$  such that

$$\lim_{n \rightarrow \infty} \|U_n^*T_1U_n - T_2\| = 0 \text{ and}$$

$$U_n^* T_1 U - T_2 \text{ is compact}$$

if and only if

- (i)  $\text{sp}_e(T_1) = \text{sp}_e(T_2)$ .
- (ii)  $\dim \text{null}(T_1 - \lambda I) = \dim \text{null}(T_2 - \lambda I)$  for all  $\lambda \in \mathbb{C} \setminus \text{sp}_e(T_1)$ .

Here  $\text{sp}_e(T_i)$  is the essential spectrum of  $T_i$ , i.e.,  $\text{sp}_e(T_i) = \text{sp}(\pi(T_i))$ , where  $\pi : B(l^2) \rightarrow B(l^2)/\mathcal{K}$  is the quotient map,  $i = 1, 2$ . Let  $X$  be a compact subset of the plane and let  $\phi_1, \phi_2 : C(X) \rightarrow B(l^2)/\mathcal{K}$  be two unital monomorphisms. In the study of essentially normal operators on the infinite dimensional separable Hilbert space, one asks when  $\phi_1$  and  $\phi_2$  are unitarily equivalent? This was answered by the celebrated Brown-Douglas-Fillmore Theorem:  $\phi_1$  and  $\phi_2$  are unitarily equivalent if and only if  $(\phi_1)_{*1} = (\phi_2)_{*1}$ , where  $(\phi_i)_{*1} : K_1(C(X)) \rightarrow K_1((B(l^2)/\mathcal{K})) \cong \mathbb{Z}$  is the induced homomorphism (Fredholm index),  $i = 1, 2$  (cf. [3]). In fact, one has the following more general BDF-theorem:

**Theorem 1.2.** *If  $X$  is a compact metric space, then  $\phi_1$  and  $\phi_2$  are unitarily equivalent if and only*

$$[\phi_1] = [\phi_2] \text{ in } KK(C(X), B(l^2)/\mathcal{K})$$

(cf. [4]).

It is known that the Calkin algebra  $B(l^2)/\mathcal{K}$  is a unital simple  $C^*$ -algebra with real rank zero. It is also purely infinite. In this paper, we will study approximate unitary equivalence in a unital separable simple stably finite  $C^*$ -algebra.

**Definition 1.3.** Let  $A$  and  $B$  be two unital  $C^*$ -algebras and let  $\phi_1, \phi_2 : A \rightarrow B$  be two homomorphisms. We say that  $\phi_1$  and  $\phi_2$  are approximately unitarily equivalent if there exists a sequence of unitaries  $\{u_n\} \subset B$  such that

$$\lim_{n \rightarrow \infty} \text{ad } u_n \circ \phi_1(a) = \phi_2(a) \text{ for all } a \in A. \quad (\text{e 1.2})$$

In definition 1.3, suppose that  $J = \ker \phi_1$ . Then  $\ker \phi_2 = J$  if  $\phi_1$  and  $\phi_2$  are approximately unitarily equivalent. Thus one may study the induced monomorphisms from  $A/I$  to  $B$  instead of homomorphisms from  $A$ . To simplify matters, we will only study monomorphisms.

We note that  $M_n$  is a unital finite dimensional simple  $C^*$ -algebra with a unique tracial state. We now replace  $A$  by an infinite dimensional simple  $C^*$ -algebra. First we consider AF-algebras, approximately finite dimensional  $C^*$ -algebras.

Let  $A$  be a unital simple AF-algebra and let  $X$  be a compact metric space. Let  $\phi_1, \phi_2 : C(X) \rightarrow A$  be two unital monomorphisms. When are  $\phi_1$  and  $\phi_2$  approximately unitarily equivalent? or, when are there unitaries  $u_n \in A$  such that

$$\lim_{n \rightarrow \infty} u_n^* \phi_1(a) u_n = \phi_2(a)$$

for all  $a \in C(X)$ ?

**1.4.** Let  $C$  be a unital stably finite  $C^*$ -algebra. Denote by  $T(C)$ , throughout this paper, the tracial state space of  $C$ .

Suppose that  $\phi_1, \phi_2 : C(X) \rightarrow A$  are two unital monomorphisms. Let  $\tau \in T(A)$  be a tracial state. Then  $\tau \circ \phi_j$  is a normalized positive linear functional ( $j = 1, 2$ ). It gives a Borel probability measure  $\mu_j$ . Furthermore, it is strictly positive in the sense that  $\mu_j(O) > 0$  for every non-empty open subset  $O \subset X$ . If  $\phi_1$  and  $\phi_2$  are approximately unitarily equivalent, then it is obvious that  $\mu_1 = \mu_2$ , or equivalently,  $\tau \circ \phi_1 = \tau \circ \phi_2$ . In fact, one has the following :

**1.5.** *Let  $X$  be a compact metric space and let  $A$  be a unital simple AF-algebra with a unique tracial state  $\tau$ . Suppose that  $\phi_1, \phi_2 : C(X) \rightarrow A$  are two unital monomorphisms. Then  $\phi_1$  and  $\phi_2$  are approximately unitarily equivalent if and only if*

$$(\phi_1)_{*0} = (\phi_2)_{*0} \text{ and } \tau \circ \phi_1 = \tau \circ \phi_2.$$

Here  $(\phi_i)_{*0}$  is induced homomorphism from  $K_0(C(X))$  into  $K_0(A)$ . Note in the case that  $X$  is connected and  $K_0(A)$  has no infinitesimal elements, i.e.,  $\tau(p) = \tau(q)$  implies  $[p] = [q]$  in  $K_0(A)$  for every projection  $p$  and  $q$ , as in the case that  $A = M_n$ , or in the case that  $A$  is a UHF-algebra, the condition  $(\phi_1)_{*0} = (\phi_2)_{*0}$  is automatically satisfied if the two measures are the same. Therefore, one may view that answer 1.5 is a generalization of 1.1.

Note also that  $K_1(A) = \{0\}$ . In general,  $\phi_j$  also gives another homomorphisms:  $(\phi_j)_{*1} : K_1(C(X)) \rightarrow K_1(A)$ ,  $j = 1, 2$ .

The above answer 1.5 follows from a much more general result which serves as a uniqueness theorem in the Elliott program of classification of amenable  $C^*$ -algebras:

**Theorem 1.6.** (Gong-Lin 1996 [16]) *Let  $X$  be a compact metric space and let  $A$  be a unital simple  $C^*$ -algebra with real rank zero, stable rank one, weakly unperforated  $K_0(A)$  and with a unique tracial state  $\tau$ . Suppose that  $\phi_1, \phi_2 : C(X) \rightarrow A$  are two unital monomorphisms. Then  $\phi_1$  and  $\phi_2$  are approximately unitarily equivalent if and only if*

$$[\phi_1] = [\phi_2] \text{ in } KL(C(X), A) \text{ and } \tau \circ \phi_1 = \tau \circ \phi_2.$$

In the case that  $K_*(C(X))$  is torsion free, the condition that  $[\phi_1] = [\phi_2]$  in  $KL(C(X), A)$  can be replaced by  $(\phi_1)_{*i} = (\phi_2)_{*i}$ , where  $(\phi_j)_{*i} : K_i(C(X)) \rightarrow K_i(A)$  ( $i = 0, 1$  and  $j = 1, 2$ ) is the induced homomorphisms.

Recall that an AH-algebra is an inductive limit of  $C^*$ -algebras with the form  $P_n M_{k(n)}(C(X_n)) P_n$ , where  $X_n$  is (not necessarily connected) finite CW complex and  $P_n$  is a projection in  $M_{k(n)}(C(X_n))$ . More recently, for the situation that  $T(A)$  has no restriction, we have the following:

**Theorem 1.7.** ([27]) *Let  $C$  be a unital AH-algebra and let  $A$  be a unital separable simple  $C^*$ -algebra with tracial rank zero. Suppose that  $\phi_1, \phi_2 : C \rightarrow A$  are two unital monomorphisms. Then  $\phi_1$  and  $\phi_2$  are approximately unitarily equivalent if and only if*

$$[\phi_1] = [\phi_2] \text{ in } KL(C, A) \text{ and } \tau \circ \phi_1 = \tau \circ \phi_2 \text{ for all } \tau \in T(A). \quad (\text{e1.3})$$

Theorem 1.7 was established in the connection with the Elliott program. Versions of 1.7 plays important roles in the Elliott theory of classification of amenable  $C^*$ -algebras. It also has application in the study of minimal dynamical systems (see [27], [36], [37], [38] and [25]). More recently, Theorem 1.7 is used to study the so-called Basic Homotopy Lemma (in simple  $C^*$ -algebras with real rank zero—see [29]) and the asymptotic unitary equivalence in simple  $C^*$ -algebras with tracial rank zero ([31]) which, in turn, plays crucial roles in the recent work of AF-embedding ([32]) and classification of amenable simple finite  $C^*$ -algebras which are *not* of finite tracial rank (see [52] and [33]). It is now clear that approximately unitary equivalence and asymptotic unitary equivalence in simple  $C^*$ -algebras with tracial rank one becomes very important and useful. Moreover, to establish a theorem about asymptotic unitary equivalence in simple  $C^*$ -algebras with tracial rank one, one has first to establish a theorem about approximately unitary equivalence which can also be used to establish required Basic Homotopy Lemmas. This is the main purpose of this paper.

A consequence of the main results of this paper may be stated as follows:

**Theorem 1.8.** *Let  $C$  be a unital AH-algebra with property (J) and let  $A$  be a unital simple  $C^*$ -algebra with  $TR(A) \leq 1$ . Suppose that  $\phi, \psi : C \rightarrow A$  are two unital monomorphisms. Then  $\phi$  and  $\psi$  are approximately unitarily equivalent if and only if*

$$[\phi] = [\psi] \text{ in } KL(C, A), \quad (\text{e1.4})$$

$$\phi_{\sharp} = \psi_{\sharp} \text{ and } \phi^{\dagger} = \psi^{\dagger}. \quad (\text{e1.5})$$

(See 2.7 and 2.12 for the definition of  $\phi^{\dagger}$  and  $\phi_{\sharp}$ , and see 11.4 for the property (J)). It should be noted that it is an approximate version of the above which actually plays the role in the subsequent papers.

The paper is organized as follows. Section 2 collects some notation and conventions which will be used throughout the paper. Section 3 contains a generalization of a theorem of Elliott, Gong and Li which will be used at the end of the paper. In section 4, we show that two approximately multiplicative completely positive linear maps from  $C(X)$  to a finite dimensional  $C^*$ -algebra are almost unitarily equivalent if they induce the same  $KK$ -information and satisfy some rigidity conditions, where  $X$  is a path connected compact metric space. These are reformulation of results in [27]. Section 5 contains a version of the so-called Basic Homotopy Lemma which is a reformulation of some results in [29]. Section 6 contains the following result. Two homomorphisms from  $C(X)$  into a finite dimensional  $C^*$ -algebra are unitarily equivalent, modulo a small homotopy, if they are close to each other and they are “very injective” in a measure theoretic sense. In section 7, by applying the Basic Homotopy Lemma in section 5, we show that two unital unitarily equivalent homomorphisms from  $C(X)$  into a finite dimensional  $C^*$ -algebra are homotopic by a nearby path, if they are also close and “very injective”, at least for some special finite CW complexes. In section 8, we establish the following. With the restriction of  $X$  as in section 7, an approximately multiplicative contractive completely positive linear map  $\phi : C(X) \rightarrow C([0, 1], M_n)$  (for any  $n$ ) is close to a homomorphism provided that the  $KK$ -map induced by  $\phi$  is consistent to a homomorphism and it is “very injective”. This is one of the main technical lemma of the paper. In fact, section 3, 4, 5, 6 and 7 are all preparation for the proof of Theorem 8.3. Section 9 contains a number of elementary results about simple  $C^*$ -algebras of tracial rank one (or less). In section 10, we present the main result (Theorem 10.8). Finally, in section 11, we present a number of variations of the main results in section 10.

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## 2 Some notation and definitions

**2.1.** Let  $X$  be a compact metric space, let  $x \in X$  and let  $a > 0$ . Denote by  $B_a(x)$  the open ball of  $X$  with radius  $a$  and center  $x$ . Let  $A$  be a unital  $C^*$ -algebra and  $\xi \in X$ . Denote by  $\pi_{\xi} : C(X) \rightarrow A$  the point-evaluation defined by  $\pi_{\xi}(f) = f(\xi) \cdot 1_A$  for all  $f \in C(X)$ .

**2.2.** Let  $A$  and  $B$  be two  $C^*$ -algebras and let  $L_1, L_2 : A \rightarrow B$  be two maps. Suppose that  $\mathcal{F} \subset A$  is a subset and  $\epsilon > 0$ . We write

$$L_1 \approx_{\epsilon} L_2 \text{ on } \mathcal{F}$$

if  $\|L_1(a) - L_2(a)\| < \epsilon$  for all  $a \in \mathcal{F}$ .

Map  $L_1$  is said to be  $\epsilon$ - $\mathcal{F}$ -multiplicative if

$$\|L_1(ab) - L_1(a)L_1(b)\| < \epsilon \text{ for all } a, b \in \mathcal{F}.$$

**2.3.** Let  $A$  be a  $C^*$ -algebra. Set  $M_\infty(A) = \cup_{n=1}^\infty M_n(A)$ .

**2.4.** Let  $A$  be a unital  $C^*$ -algebra. Denote by  $U(A)$  the unitary group of  $A$ . Denote by  $U_0(A)$  the normal subgroup of  $U(A)$  consisting of the path connected component of  $U(A)$  containing the identity. Suppose that  $u \in U_0(A)$  and  $\{u(t) : t \in [0, 1]\}$  is a continuous path with  $u(0) = u$  and  $u(1) = 1$ . Denote by  $\text{length}(\{u(t)\})$  the length of the path. Put

$$\text{cel}(u) = \inf\{\text{length}(\{u(t)\})\}.$$

**Definition 2.5.** Let  $X$  be a compact metric space and let  $P \in M_l(C(X))$  be a projection. Put  $C = PM_l(C(X))P$ . Let  $u \in U(C)$ . Define, as in [42],

$$D_C(u) = \inf\{\|a\| : a \in A_{s,a} \text{ such that } \det(e^{ia} \cdot u) = 1\}. \quad (\text{e2.6})$$

**2.6.** Let  $A$  be a unital  $C^*$ -algebra. Denote by  $CU(A)$  the *closure* of the subgroup generated by the commutators of  $U(A)$ . For  $u \in U(A)$ , we will use  $\bar{u}$  for the image of  $u$  in  $U(A)/CU(A)$ .

If  $\bar{u}, \bar{v} \in U(A)/CU(A)$ , define

$$\text{dist}(\bar{u}, \bar{v}) = \inf\{\|x - y\| : x, y \in U(A) \text{ such that } \bar{x} = \bar{u}, \bar{y} = \bar{v}\}.$$

If  $u, v \in U(A)$ , then

$$\text{dist}(\bar{u}, \bar{v}) = \inf\{\|uv^* - x\| : x \in CU(A)\}.$$

**2.7.** Let  $A$  and  $B$  be two unital  $C^*$ -algebras and let  $\phi : A \rightarrow B$  be a unital homomorphism. It is easy to check that  $\phi$  maps  $CU(A)$  to  $CU(B)$ . Denote by  $\phi^\dagger$  the homomorphism from  $U(A)/CU(A)$  into  $U(B)/CU(B)$  induced by  $\phi$ . We also use  $\phi^\dagger$  for the homomorphism from  $U(M_k(A))/CU(M_k(A))$  into  $U(M_k(B))/CU(M_k(B))$  ( $k = 1, 2, \dots$ ).

**Definition 2.8.** Let  $A$  be a  $C^*$ -algebra. Following Dadarlat and Loring ([7]), denote

$$\underline{K}(A) = \oplus_{i=0,1} K_i(A) \bigoplus_{i=0,1} \bigoplus_{k \geq 2} K_i(A, \mathbb{Z}/k\mathbb{Z}).$$

Let  $B$  be a unital  $C^*$ -algebra. If furthermore,  $A$  is assumed to be separable and satisfy the Universal Coefficient Theorem ([45]), by [7],

$$\text{Hom}_\Lambda(\underline{K}(A), \underline{K}(B)) = KL(A, B).$$

Here  $KL(A, B) = KK(A, B)/\text{Pext}(K_*(A), K_*(B))$  (see [7] for details).

Let  $k \geq 1$  be an integer. Denote

$$F_k \underline{K}(A) = \oplus_{i=0,1} K_i(A) \bigoplus_{n|k} K_i(A, \mathbb{Z}/k\mathbb{Z}).$$

Suppose that  $K_i(A)$  is finitely generated ( $i = 0, 2$ ). It follows from [7] that there is an integer  $k \geq 1$  such that

$$\text{Hom}_\Lambda(F_k \underline{K}(A), F_k \underline{K}(B)) = \text{Hom}_\Lambda(\underline{K}(A), \underline{K}(B)). \quad (\text{e2.7})$$

**2.9.** Let  $A$  and  $B$  be two unital  $C^*$ -algebras and let  $L : A \rightarrow B$  be a unital contractive completely positive linear map. Let  $\mathcal{P} \subset \underline{K}(A)$  be a finite subset. It is well known that, for some small  $\delta$  and large finite subset  $\mathcal{G} \subset A$ , if  $L$  is also  $\delta$ - $\mathcal{G}$ -multiplicative, then  $[L]|_{\mathcal{P}}$  is well defined. In what follows whenever we write  $[L]|_{\mathcal{P}}$  we mean  $\delta$  is sufficiently small and  $\mathcal{G}$  is sufficiently large so that it is well defined (see 2.3 of [29]). If  $u \in U(A)$ , we will use  $\langle L \rangle(u)$  for the unitary  $L(u)|L(u)^*L(u)|^{-1}$ .

For an integer  $m \geq 1$  and a finite subset  $\mathcal{U} \subset U(M_m(A))$ , let  $F \subset U(A)$  be the subgroup generated by  $\mathcal{U}$ . As in 6.2 of [26], there exists a finite subset  $\mathcal{G}$  and a small  $\delta > 0$  such that a  $\delta$ - $\mathcal{G}$ -multiplicative contractive completely positive linear map  $L$  induces a homomorphism  $L^\dagger : \overline{F} \rightarrow U(M_m(B))/CU(M_m(B))$ . Moreover, we may assume,  $\langle L \rangle(\bar{u}) = L^\dagger(\bar{u})$ .

If there are  $L_1, L_2 : A \rightarrow B$  and  $\epsilon > 0$  is given. Suppose that both  $L_1$  and  $L_2$  are  $\delta$ - $\mathcal{G}$ -multiplicative and  $L_1^\dagger$  and  $L_2^\dagger$  are well defined on  $\overline{F}$ , whenever, we write

$$\text{dist}(L_1^\dagger(\bar{u}), L_2^\dagger(\bar{u})) < \epsilon$$

for all  $u \in \mathcal{U}$ , we also assume that  $\delta$  is sufficiently small and  $\mathcal{G}$  is sufficiently large so that

$$\text{dist}(\overline{\langle L_1 \rangle(u)}, \overline{\langle L_2 \rangle(u)}) < \epsilon \text{ for all } u \in \mathcal{U}.$$

**Definition 2.10.** Let  $A$  and  $B$  be two unital  $C^*$ -algebras. Let  $h : A \rightarrow B$  be a homomorphism and  $v \in U(B)$  such that

$$h(g)v = vh(g) \text{ for all } g \in A.$$

Thus we obtain a homomorphism  $\bar{h} : A \otimes C(\mathbb{T}) \rightarrow B$  by  $\bar{h}(f \otimes g) = h(f)g(v)$  for  $f \in A$  and  $g \in C(\mathbb{T})$ . The tensor product induces two injective homomorphisms:

$$\beta^{(0)} : K_0(A) \rightarrow K_1(A \otimes C(\mathbb{T})) \text{ and} \quad (\text{e 2.8})$$

$$\beta^{(1)} : K_1(A) \rightarrow K_0(A \otimes C(\mathbb{T})). \quad (\text{e 2.9})$$

The second one is the usual Bott map. Note, in this way, one writes

$$K_i(A \otimes C(\mathbb{T})) = K_i(A) \oplus \beta^{(i-1)}(K_{i-1}(A)).$$

We use  $\widehat{\beta^{(i)}} : K_i(A \otimes C(\mathbb{T})) \rightarrow \beta^{(i-1)}(K_{i-1}(A))$  for the projection to  $\beta^{(i-1)}(K_{i-1}(A))$ .

For each integer  $k \geq 2$ , one also obtains the following injective homomorphisms:

$$\beta_k^{(i)} : K_i(A, \mathbb{Z}/k\mathbb{Z}) \rightarrow K_{i-1}(A \otimes C(\mathbb{T}), \mathbb{Z}/k\mathbb{Z}), i = 0, 1. \quad (\text{e 2.10})$$

Thus we write

$$K_{i-1}(A \otimes C(\mathbb{T}), \mathbb{Z}/k\mathbb{Z}) = K_{i-1}(A, \mathbb{Z}/k\mathbb{Z}) \oplus \beta_k^{(i)}(K_i(A, \mathbb{Z}/k\mathbb{Z})), i = 0, 1. \quad (\text{e 2.11})$$

Denote by  $\widehat{\beta_k^{(i)}} : K_i(A \otimes C(\mathbb{T}), \mathbb{Z}/k\mathbb{Z}) \rightarrow \beta_k^{(i-1)}(K_{i-1}(A, \mathbb{Z}/k\mathbb{Z}))$  similarly to that of  $\widehat{\beta^{(i)}}$ ,  $i = 1, 2$ . If  $x \in \underline{K}(A)$ , we use  $\beta(x)$  for  $\beta^{(i)}(x)$  if  $x \in K_i(A)$  and for  $\beta_k^{(i)}(x)$  if  $x \in K_i(A, \mathbb{Z}/k\mathbb{Z})$ . Thus we have a map  $\beta : \underline{K}(A) \rightarrow \underline{K}(A \otimes C(\mathbb{T}))$  as well as  $\widehat{\beta} : \underline{K}(A \otimes C(\mathbb{T})) \rightarrow \beta(\underline{K}(A))$ . Therefore one may write  $\underline{K}(A \otimes C(\mathbb{T})) = \underline{K}(A) \oplus \beta(\underline{K}(A))$ . On the other hand  $\bar{h}$  induces homomorphisms  $\bar{h}_{*,k} : K_i(A \otimes C(\mathbb{T}), \mathbb{Z}/k\mathbb{Z}) \rightarrow K_i(B, \mathbb{Z}/k\mathbb{Z})$ ,  $k = 0, 2, \dots$ , and  $i = 0, 1$ .

We use  $\text{Bott}(h, v)$  for all homomorphisms  $\bar{h}_{*,k} \circ \beta_k^{(i)}$ . We write

$$\text{Bott}(h, v) = 0,$$

if  $\bar{h}_{*i,k} \circ \beta_k^{(i)} = 0$  for all  $k \geq 1$  and  $i = 0, 1$ . We will use  $\text{bott}_1(h, v)$  for the homomorphism  $\bar{h}_{1,0} \circ \beta^{(1)} : K_1(A) \rightarrow K_0(B)$ , and  $\text{bott}_0(h, u)$  for the homomorphism  $\bar{h}_{0,0} \circ \beta^{(0)} : K_0(A) \rightarrow K_1(B)$ . Since  $A$  is unital, if  $\text{bott}_0(h, v) = 0$ , then  $[v] = 0$  in  $K_1(B)$ .

In what follows, we will use  $z$  for the standard generator of  $C(\mathbb{T})$  and we will often identify  $\mathbb{T}$  with the unit circle without further explanation. With this identification  $z$  is the identity map from the circle to the circle.

**2.11.** Given a finite subset  $\mathcal{P} \subset \underline{K}(A)$ , there exists a finite subset  $\mathcal{F} \subset A$  and  $\delta_0 > 0$  such that

$$\text{Bott}(h, v)|_{\mathcal{P}}$$

is well defined, if

$$\|[h(a), v]\| = \|h(a)v - vh(a)\| < \delta_0 \text{ for all } a \in \mathcal{F}$$

(see 2.10 of [29]). There is  $\delta_1 > 0$  ([39]) such that  $\text{bott}_1(u, v)$  is well defined for any pair of unitaries  $u$  and  $v$  such that  $\|[u, v]\| < \delta_1$ . As in 2.2 of [13], if  $v_1, v_2, \dots, v_n$  are unitaries such that

$$\|[u, v_j]\| < \delta_1/n, \quad j = 1, 2, \dots, n,$$

then

$$\text{bott}_1(u, v_1 v_2 \cdots v_n) = \sum_{j=1}^n \text{bott}_1(u, v_j).$$

By considering unitaries  $z \in \widetilde{A \otimes C}$  ( $C = C_n$  for some commutative  $C^*$ -algebra with torsion  $K_0$  and  $C = SC_n$ ), from the above, for a given unital  $C^*$ -algebra  $A$  and a given finite subset  $\mathcal{P} \subset \underline{K}(A)$ , one obtains a universal constant  $\delta > 0$  and a finite subset  $\mathcal{F} \subset A$  satisfying the following:

$$\text{Bott}(h, v_j)|_{\mathcal{P}} \text{ is well defined and } \text{Bott}(h, v_1 v_2 \cdots v_n) = \sum_{j=1}^n \text{Bott}(h, v_j), \quad (\text{e 2.12})$$

for any unital homomorphism  $h$  and unitaries  $v_1, v_2, \dots, v_n$  for which

$$\|[h(a), v_j]\| < \delta/n, \quad j = 1, 2, \dots, n, \text{ for all } a \in \mathcal{F}. \quad (\text{e 2.13})$$

If furthermore,  $K_i(A)$  is finitely generated, then (e 2.7) holds. Therefore, there is a finite subset  $\mathcal{Q} \subset \underline{K}(A)$ , such that

$$\text{Bott}(h, v)$$

is well defined if  $\text{Bott}(h, v)|_{\mathcal{Q}}$  is well defined (see also 2.3 of [29]). See Section 2 of [29] for the further information.

**2.12.** Let  $A$  be a unital  $C^*$ -algebra. Denote by  $T(A)$  the tracial state space of  $A$ . Suppose that  $T(A) \neq \emptyset$ . Let  $B$  be another unital  $C^*$ -algebra with  $T(B) \neq \emptyset$ . Suppose that  $\phi : A \rightarrow B$  is a unital homomorphism. Denote by  $\phi_{\#} : \text{Aff}(T(A)) \rightarrow \text{Aff}(T(B))$  the positive homomorphism defined by  $\phi_{\#}(\hat{a})(\tau) = \tau \circ \phi(a)$  for all  $a \in A_{s.a.}$ .

**2.13.** Let  $X$  be a compact metric space and let  $A$  be a unital  $C^*$ -algebra with  $T(A) \neq \emptyset$ . Let  $L : C(X) \rightarrow A$  be a unital positive linear map. For each  $\tau \in T(A)$  denote by  $\mu_{\tau \circ L}$  the Borel probability measure induced by  $\tau \circ L$ .

**2.14.** Let  $X_1, X_2, \dots, X_m$  be compact metric spaces. Fix a base point  $\xi_i \in X_i$ ,  $i = 1, 2, \dots, m$ . We write  $X_1 \vee X_2 \vee \dots \vee X_m$  the space resulted by gluing  $X_1, X_2, \dots, X_m$  together at  $\xi_i$  (by identifying all base points at one point  $\xi_1$ ). Denote by  $\xi_0$  the common point. If  $x, y \in X_i$ , then  $\text{dist}(x, y)$  is defined to be the same as that in  $X_i$ . If  $x \in X_i, y \in X_j$  with  $i \neq j$ , and  $x \neq \xi_0, y \neq \xi_0$ , then we define

$$\text{dist}(x, y) = \text{dist}(x, \xi_0) + \text{dist}(y, \xi_0).$$

**Definition 2.15.** ([21]) Let  $A$  be a unital simple  $C^*$ -algebra.  $A$  is said to have tracial rank no more than one ( $TR(A) \leq 1$ ) if the following hold: For any  $\epsilon > 0$ , any  $a \in A_+ \setminus \{0\}$  and any finite subset  $\mathcal{F} \subset A$  there exists a projection  $p \in A$  and a  $C^*$ -subalgebra  $B = \oplus_{i=1}^k M_{r(i)}(C(X_i))$ , where each  $X_i$  is a finite CW complex with covering dimension no more than 1, with  $1_B = p$  such that

- (1)  $\|px - xp\| < \epsilon$  for all  $x \in \mathcal{F}$ ,
- (2)  $\text{dist}(pxp, B) < \epsilon$  for all  $\mathcal{F}$  and
- (3)  $1 - p$  is equivalent to a projection in  $\overline{aAa}$ .

If in the above definition,  $X_i$  can always be chosen to be a point, then we say  $A$  has tracial rank zero and write  $TR(A) = 0$ . If  $TR(A) \leq 1$  but  $TR(A) \neq 0$ , then we write  $TR(A) = 1$  and say  $A$  has tracial rank one. By 7.1 of [21], if  $TR(A) \leq 1$ , then  $A$  has TAI, i.e., in the above definition, one may replace  $X_i$  by  $[0, 1]$  or by a point.

**2.16.** Let  $A$  be a unital separable simple  $C^*$ -algebra with  $TR(A) \leq 1$ . Then  $A$  is tracially approximately divisible: i.e., for any  $\epsilon > 0$ , any finite subset  $\mathcal{F} \subset A$ , any  $a \in A_+ \setminus \{0\}$  and any integer  $N \geq 1$ , there exists a projection  $p \in A$  and a finite dimensional  $C^*$ -subalgebra  $D = \oplus_{i=1}^k M_{r(i)}$  with  $r(j) \geq N$  and with  $1_D = p$  such that

- (1)  $\|[x, y]\| < \epsilon$  for all  $x \in \mathcal{F}$  and for all  $y \in D$  with  $\|y\| \leq 1$ ;
- (2)  $1 - p$  is equivalent to a projection in  $\overline{aAa}$

(see 5.4 of [26]).

### 3 A uniqueness theorem

This section will not be used until the proof of 10.8.

**Lemma 3.1.** (Proposition 4.47' of [14]) *Let  $X$  be a connected simplicial complex, let  $\mathcal{F} \subset C(X)$  be a finite subset, let  $\epsilon > 0$ ,  $\epsilon_1 > 0$  be positive numbers, and let  $N \geq 1$  be an integer. There exists  $\eta_1 > 0$  with the following properties.*

*For any  $\sigma_1 > 0$  and any  $\sigma > 0$ , there exists a positive number  $\eta > 0$  and an integer  $K > 4/\epsilon$  (which are independent of  $\sigma$ ), there exists a positive number  $\delta > 0$ , an integer  $L > 0$  and a finite subset  $\mathcal{G} \subset C(X)$  satisfying the following:*

*Suppose that  $\phi, \psi : C(X) \rightarrow PM_n(C(Y))P$  (where  $Y$  is a connected simplicial complex with  $\dim Y \leq 3$ ), where  $\text{rank}(P) \geq L$ , are two unital homomorphisms such that*

$$\mu_{\tau \circ \phi}(O_{\eta_1}) \geq \sigma_1 \eta_1, \mu_{\tau \circ \psi}(O_{\eta_1}) \geq \sigma \eta \text{ for all } \tau \in T(PM_n(C(X))P) \quad (\text{e3.14})$$

*and for all open balls  $O_{\eta_1}$  with radius  $\eta_1$  and open balls  $O_{\eta_2}$  with radius  $\eta_2$ , respectively, and*

$$|\tau \circ \phi(g) - \tau \circ \psi(g)| < \delta \text{ for all } g \in \mathcal{G}. \quad (\text{e3.15})$$



Then there exist mutually orthogonal projections  $P_0$  and  $P_1$  (with  $P_0 + P_1 = P$ ), a unital homomorphism  $\phi_1 : C(X) \rightarrow P_1(M_n(C(Y)))P_1$  factoring through  $C([0, 1])$ , and a unitary  $u \in P(M_n(C(Y)))P$  such that

$$\|\phi(f) - [P_0\phi(f)P_0 + \phi_1(f)]\| < 1/4K \text{ and} \quad (\text{e 3.16})$$

$$\|\text{ad } u \circ \psi(f) - [P_0(\text{ad } u \circ \psi(f))P_0 + \phi_1(f)]\| < 1/4K \text{ for all } f \in \mathcal{F}, \quad (\text{e 3.17})$$

$$\text{rank} P_0 \geq \frac{\text{rank} P}{K}, \quad (\text{e 3.18})$$

there are mutually orthogonal projections  $q_1, q_2, \dots, q_m \in P_1(M_n(C(Y)))P_1$  and an  $\epsilon_1$ -dense subset  $\{x_1, x_2, \dots, x_m\}$  such that

$$\|\phi_1(f) - [(P_1 - \sum_{j=1}^m q_j)\phi_1(f)(P_1 - \sum_{j=1}^m q_j) + \sum_{j=1}^m f(x_j)q_j]\| < \epsilon \quad (\text{e 3.19})$$

for all  $f \in \mathcal{F}$  and

$$\text{rank}(q_j) \geq N \cdot (\text{rank} P_0 + 2\dim Y), \quad j = 1, 2, \dots, m. \quad (\text{e 3.20})$$

*Proof.* This is a reformulation of Proposition 4.47' of [14] and follows from that immediately.

We now will apply Proposition 4.47' of [14]. Let  $\epsilon > 0$ ,  $\epsilon_1 > 0$ ,  $N$  and  $\mathcal{F}$  be given. Choose  $\eta_0 > 0$  such that

$$|f(x) - f(x')| < \epsilon/2 \text{ for all } f \in \mathcal{F}. \quad (\text{e 3.21})$$

Choose  $\epsilon_2 = \min\{\epsilon_1/3N, \eta_0/3N\}$ . Let  $\eta'_1 > 0$  (in place of  $\eta$ ) be as in Proposition 4.47' of [14] for  $\epsilon/2$ ,  $\epsilon_2$  (in place of  $\epsilon_1$ ) and  $\mathcal{F}$ . Let  $\sigma_1 > 0$  and  $\sigma > 0$ . Put  $\delta_1 = \sigma_1 \cdot \eta'_1/32$ . Let  $K > 4/\epsilon$  and  $\tilde{\eta}$  be as in Proposition 4.47' of [14] for the above  $\epsilon/4$ ,  $\epsilon_2$  (in place of  $\epsilon_1$ ) and  $\delta_1$  (in place of  $\delta$ ). Let  $\tilde{\delta} = \sigma \cdot \tilde{\eta}/32$ . Let  $L \geq 1$  be an integer and let  $\mathcal{G} \subset C(X)$  be a finite subset which corresponds the finite subset  $H$  in Proposition 4.47' of [14]. Let  $\eta_1 = \eta'_1/32$ ,  $\eta = \tilde{\eta}/32$  and let  $0 < \delta < \tilde{\delta}/4$ . Suppose that  $\phi$  and  $\psi$  satisfy the assumption of the lemma for the above  $\eta_1$ ,  $\eta$ ,  $\delta$ ,  $K$ ,  $L$  and  $\mathcal{G}$ .

It follows that  $\phi$  has the properties  $\text{sdp}(\eta_1/32, \delta_1)$  and  $\text{sdp}(\tilde{\eta}/32, \tilde{\delta})$  (see 2.1 of [14]). One then applies Proposition 4.47' of [14] to obtain

$$\|\phi(f) - [P_0\phi(f)P_0 + \phi_1(f)]\| < 1/4K \text{ and} \quad (\text{e 3.22})$$

$$\|\text{ad } u \circ \psi(f) - [P_0(\text{ad } u \circ \psi(f))P_0 + \phi_1(f)]\| < 1/4K \text{ for all } f \in \mathcal{F}, \quad (\text{e 3.23})$$

and mutually orthogonal projections  $e_1, e_2, \dots, e_{m_1}$  in  $P_1(M_n(C(Y)))P_1$  and  $\epsilon_2/4$ -dense subset  $\{x'_1, x'_2, \dots, x'_{m_1}\}$  of  $X$  such that

$$\|\phi_1(f) - [(P_1 - \sum_{i=1}^{m_1} e_i)\phi_1(f)(P_1 - \sum_{i=1}^{m_1} e_i) + \sum_{i=1}^{m_1} f(x'_i)e_i]\| < \epsilon/2 \quad (\text{e 3.24})$$

for all  $f \in \mathcal{F}$ ,

$$\text{rank} P_0 \geq \frac{\text{rank} P}{K} \text{ and } \text{rank} e_i \geq \text{rank} P_0 + 2\dim Y. \quad (\text{e 3.25})$$

Since there are at least  $N$  many disjoint open balls with radius  $\epsilon_2$  in an open ball of radius  $\epsilon_1$ , by moving points within  $N\epsilon_2 < \min\{\epsilon_1/2, \eta_0\}$ , by (e 3.21), one may write

$$\|\phi_1(f) - [(P_1 - \sum_{i=1}^{m_1} e_i)\phi_1(f)(P_1 - \sum_{i=1}^{m_1} e_i) + \sum_{i=1}^{m_1} f(x_i)q_i]\| < \epsilon \quad (\text{e 3.26})$$

for all  $f \in \mathcal{F}$  and

$$\text{rank} q_i \geq N(\text{rank} P_0 + 2\dim Y), \quad (\text{e 3.27})$$

where  $\sum_{i=1}^m q_i = \sum_{i=1}^{m_1} e_i$ .

□

The following is a generalization of Theorem 2.11 of [11]. The proof is essentially the same but we will also apply [17].

**Theorem 3.2.** (cf. Theorem 2.11 of [11]) *Let  $X$  be a finite simplicial complex, let  $\mathcal{F} \subset C(X)$  be a finite subset and let  $\epsilon > 0$ . There exists  $\eta_1 > 0$  with the following property.*

*For any  $\sigma_1 > 0$  and  $\sigma > 0$ , there exists  $\eta > 0$  and an integer  $K$  (which are independent of  $\sigma$ ), there exists  $\delta > 0$ , a finite subset  $\mathcal{G} \subset C(X)$ , a finite subset  $\mathcal{P} \subset \underline{K}(C(X))$ , a finite subset  $\mathcal{U} \subset \mathbf{P}^{(1)}(C(X))$  and a positive integer  $L$  satisfying the following:*

*Suppose that  $\phi, \psi : C(X) \rightarrow PM_k(C(Y))P$ , where  $Y$  is a connected simplicial complex with  $\dim Y \leq 3$ , are two unital homomorphisms such that*

$$\mu_{\tau \circ \phi}(O_{\eta_1}) \geq \sigma_1 \eta_1 \text{ and } \mu_{\tau \circ \psi}(O_\eta) \geq \sigma \eta \quad (\text{e 3.28})$$

*for all open balls  $O_{\eta_1}$  with radius  $\eta_1$  and open balls  $O_\eta$  with radius  $\eta$ , and*

$$|\tau \circ \phi(g) - \tau \circ \psi(g)| < \delta \text{ for all } g \in \mathcal{G} \quad (\text{e 3.29})$$

*and for all  $\tau \in T(PM_k(C(Y))P)$ ,*

$$\text{rank}(P) \geq L, \quad (\text{e 3.30})$$

$$[\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}} \text{ and} \quad (\text{e 3.31})$$

$$\text{dist}(\phi^\dagger(\bar{z}), \psi^\dagger(\bar{z})) < 1/8K\pi \quad (\text{e 3.32})$$

*for all  $z \in \mathcal{U}$ . Then there exists a unitary  $u \in PM_k(C(X))P$  such that*

$$\|\phi(f) - \text{ad } u \circ \psi(f)\| < \epsilon \text{ for all } f \in \mathcal{F}. \quad (\text{e 3.33})$$

*Proof.* It is clear that we may assume that  $X$  is connected. Since  $X$  is a simplicial simplex, there is  $k_0 \geq 1$  such that for any unital separable  $C^*$ -algebra  $A$ ,

$$\text{Hom}_\Lambda(\underline{K}(C(X)), \underline{K}(A)) = \text{Hom}_\Lambda(F_{k_0} \underline{K}(C(X)), F_{k_0} \underline{K}(A))$$

(see [7]).

Let  $C_j$  be a commutative  $C^*$ -algebra with  $K_0(C_j) = \mathbb{Z}/j\mathbb{Z}$  and  $K_1(C_j) = \{0\}$ ,  $j = 1, 2, \dots, k_0$ . Put  $D_0 = C(X)$  and  $D_j = (C(X) \otimes C_j)^\sim$ ,  $j = 1, 2, \dots, k_0$ . There is an integer  $m_1 \geq 1$  such that  $U(M_{m_1}(D_j))/U_0(M_{m_1}(D_j)) = K_1(D_j)$ ,  $j = 0, 1, 2, \dots, k_0$ . Put  $N_1 = (m_1)^2$ . Let  $r : \mathbb{N} \rightarrow \mathbb{N}$  such that  $r(n) = 3k_0n$ . Let  $b : U(M_\infty(C(X))) \rightarrow \mathbb{R}_+$  be defined by  $b(u) = (8 + 2N_1)\pi$ .

Let  $\epsilon > 0$  and  $\mathcal{F}$  be given. We may assume, without loss of generality, that  $\mathcal{F}$  is in the unit ball of  $C(X)$ . Let  $1 > \delta_1 > 0$  (in place of  $\delta$ ), let  $\mathcal{G}_1 \subset C(X)$ , let  $l \geq 1$  be an integer, let  $\mathcal{P}_0 \subset \mathbf{P}^{(0)}(C(X))$  and let  $\mathcal{U} \subset \mathbf{P}^{(1)}(C(X))$  be as required by Theorem 1.1 of [17] for  $\epsilon/4$  and  $\mathcal{F}$  (and for the above  $r$  and  $b$ ). We may assume that  $\mathcal{U} \subset \cup_{j=0}^{k_0} M_{m_1}(D_j)$ . We may also assume that there is  $l_1 \geq 1$  such that  $\mathcal{P}_0 \subset \cup_{j=0}^{k_0} M_{l_1}(D_j)$ .

We also assume that, for any unital  $C^*$ -algebra  $A$ , if  $u$  is a unitary and  $e$  is a projection for which

$$\|eu - ue\| < \delta',$$

there is a unitary  $v \in eAe$  such that

$$\|eue - v\| < 2\delta'$$

for any  $0 < \delta' < \delta_1$ .

Set  $\mathcal{F}_1 = \mathcal{F} \cup \mathcal{G}_1$ . Let  $\epsilon_1 > 0$  be such that

$$|f(x) - f(x')| < \epsilon/4 \text{ for all } f \in \mathcal{F}_1, \quad (\text{e 3.34})$$

if  $\text{dist}(x, x') < \epsilon_1$ .

Put  $N = l + 1$  and  $\epsilon_2 = \min\{\delta_1/4, \epsilon/4\}$ . Let  $\eta_1 > 0$  be required by 3.1 for  $\epsilon/2$  (in place of  $\epsilon$ ),  $\epsilon_1$ ,  $\mathcal{F}_1$  (in place of  $\mathcal{F}$ ) and  $N$ . Fix  $\sigma_1 > 0$ . Let  $\eta > 0$  and  $K_1 > 4N_1/\epsilon_2$  (in place of  $K$ ) be required by 3.1. Fix  $\sigma > 0$ . Let  $\delta > 0$ , an integer  $L > 0$  and let  $\mathcal{G} \subset C(X)$  be a finite subset required by 3.1 for  $\epsilon_2$  (in place of  $\epsilon$ ),  $\mathcal{F}_1$  (in place of  $\mathcal{F}$ ),  $\sigma$ ,  $\sigma_1$ , and  $N$ .

We may assume that  $\mathcal{G} \supset \mathcal{F}_1$ . Let  $\mathcal{P} \subset \underline{K}(C(X))$  be a finite subset which consists of the image of  $\mathcal{P}_0$  and image of  $\mathcal{U}$  in  $\underline{K}(C(X))$ , and let  $K = 2N_1K_1$ .

Now suppose that  $\phi, \psi : C(X) \rightarrow PM_k(C(Y))P$  are unital homomorphisms such that (e 3.28), (e 3.29), (e 3.30), (e 3.31) and (e 3.32) hold. It follows from 3.1 that there are mutually orthogonal projections  $P_0$  and  $P_1$  with  $P_0 + P_1 = P$ , a unital homomorphism  $\phi_1 : C(X) \rightarrow P_1(M_n(C(Y))P_1)$  factoring through  $C([0, 1])$ , and a unitary  $v \in P(M_n(C(Y)))P$  such that

$$\|\phi(f) - [P_0\phi(f)P_0 + \phi_1(f)]\| < 1/4K_1 \text{ and} \quad (\text{e 3.35})$$

$$\|\text{ad } v \circ \psi(f) - [P_0(\text{ad } v \circ \psi(f))P_0 + \phi_1(f)]\| < 1/4K_1 \text{ for all } f \in \mathcal{F}_1, \quad (\text{e 3.36})$$

$$\text{rank } P_0 \geq \frac{\text{rank } P}{K_1}, \quad (\text{e 3.37})$$

there are mutually orthogonal projections  $q_1, q_2, \dots, q_m \in P_1(M_n(C(Y)))P_1$  and an  $\epsilon_1$ -dense subset  $\{x_1, x_2, \dots, x_m\}$  such that

$$\|\phi_1(f) - [(P_1 - \sum_{j=1}^m q_j)\phi_1(f)(P_1 - \sum_{j=1}^m q_j) + \sum_{j=1}^m f(x_j)q_j]\| < \epsilon_2 \quad (\text{e 3.38})$$

for all  $f \in \mathcal{F}_1$  and

$$\text{rank}(q_j) \geq N(\text{rank } P_0 + 2\dim Y), \quad j = 1, 2, \dots, m. \quad (\text{e 3.39})$$

Note that  $1/4K_1 < \delta_1/16(N_1)$ . For each  $C_j$ , we may assume that  $C_j = C_0(Z_j \setminus \{\xi_j\})$ , where  $Z_j$  is a path connected CW complex with  $K_0(Z_j) = \mathbb{Z} \oplus \mathbb{Z}/j\mathbb{Z}$  and  $K_1(Z_j) = \{0\}$  and  $\xi_j \in Z_j$  is a point.  $j = 1, 2, \dots, k_0$ .

For each  $z \in \mathcal{U}$  and  $z \in M_{m_1}(D_j)$ , denote by  $z_1 = (\tilde{\phi} \otimes \text{id}_{m_1})(z)$  and  $z_2 = (\widetilde{\text{ad } v \circ \psi}) \otimes \text{id}_{m_1}(z)$ , where  $\tilde{\phi}, \widetilde{\text{ad } v \circ \psi} : D_j \rightarrow (C(Y) \otimes C_j)$  is the induced homomorphism.

Identifying  $(M_k(C(Y) \otimes C_j))$  with a  $C^*$ -subalgebra of  $C(Z_j, M_k(C(Y)))$  and denote by  $P'_0$  the constant projection which is  $P_0$  at each point of  $Z_j$  and denote by  $P'$  the constant projection which is  $P$  at each point of  $Z_j$ . There are unitaries  $z'_1, z'_2 \in M_{m_1}(P'_0 M_k((C(Y) \otimes C_j) P'_0))$  such that

$$\|z'_1 - \bar{P}_0 z_1 \bar{P}_0\| < \frac{2N_1}{4K_1} < \delta_1/8, \quad \|z'_2 - \bar{P}_0 z_2 \bar{P}_0\| < \frac{2N_1}{4K_1} < \delta_1/8, \quad (\text{e 3.40})$$

$$\|z_1 - z'_1 \oplus \phi_1(z)\| < \frac{3(N_1)^2}{4K_1} < \delta_1/4 \text{ and } \|z_2 - z'_2 \oplus \phi_1(z)\| < \frac{3(N_1)^2}{4K_1} < \delta_1/4, \quad (\text{e 3.41})$$

where  $\bar{P} = \text{diag}(\overbrace{P', P', \dots, P'}^{m_1})$  and  $\bar{P}_0 = \text{diag}(\overbrace{P'_0, P'_0, \dots, P'_0}^{m_1})$ . By (e 3.32), one computes that

$$\text{dist}(\overline{z'_1 \oplus \phi_1(z)}, \overline{z'_2 \oplus \phi_1(z)}) \leq \frac{1}{4K\pi} + \frac{6N_1}{4K_1} < \frac{1 + 6N_1^2\pi}{4N_1K_1\pi}, \quad (\text{e 3.42})$$

where  $\overline{z'_1 \oplus \phi_1(z)}$  and  $\overline{z'_2 \oplus \phi_1(z)}$  are the images of  $z'_1 \oplus \phi_1(z)$  and  $z'_2 \oplus \phi_1(z)$ . It follows that

$$D((z'_1(z'_2)^* \oplus (\bar{P} - \bar{P}_0))) < \frac{1 + 6N_1^2\pi}{4N_1K_1}, \quad (\text{e 3.43})$$

where  $D$  is the determinant defined in 2.5.

Since  $\text{rank} P_0 \geq \frac{\text{rank} P}{K_1}$ , by Lemma 3.3 (2) of [42],

$$D_{P_0 M_k(C(Y)) P_0}(z'_1(z'_2)^*) \leq \frac{1 + 6N_1^2\pi}{4N_1}. \quad (\text{e 3.44})$$

By the choice of  $\mathcal{P}$  and the assumption (e 3.31), since  $\dim Y \leq 3$ ,

$$z'_1(z'_2)^* \oplus \text{diag}(\overbrace{P'_0, P'_0, \dots, P'_0}^{3k_0 m_1}) \in U_0(M_{3k_0 m_1}(P'_0 M_k(D_j) P'_0)). \quad (\text{e 3.45})$$

By 3.4 of [42],

$$\text{cel}(z'_1(z'_2)^* \oplus \text{diag}(\overbrace{P'_0, P'_0, \dots, P'_0}^{3k_0 m_1})) \leq (2N_1\pi + \pi) + 6\pi \leq (2N_1 + 7)\pi \quad (\text{e 3.46})$$

for all  $z \in \mathcal{U}$ . Denote by  $\phi' = P_0 \phi P_0$  and  $\psi' = P_0(\text{ad } u \circ \psi) P_0$ . Then both are  $\delta_1$ - $\mathcal{F}_1$ -multiplicative. By the assumption (e 3.31),

$$[\phi']|_{\mathcal{P}} = [\psi']|_{\mathcal{P}}. \quad (\text{e 3.47})$$

Since  $\dim Y \leq 3$ , for any  $p \in \mathcal{P}_0$ , it follows that

$$[\phi'](p) \oplus \text{diag}(\overbrace{P_0, P_0, \dots, P_0}^{3k_0 l_1}) \sim [\psi'](p) \oplus \text{diag}(\overbrace{P_0, P_0, \dots, P_0}^{3k_0 l_1}) \quad (\text{e 3.48})$$

for all  $p \in \mathcal{P}_0$ . Note that  $3k_0 l_1 = r(l_1)$  and  $(2N_1 + 7)\pi + \delta_1/4 < b(z)$  for any  $z$ . Since (e 3.39) holds,  $N \geq l$  and  $\{x_1, x_2, \dots, x_m\}$  is  $\epsilon_1$ -dense in  $X$ , by Theorem 1.1 (and its remark) of [17], there exists a unitary  $u_1 \in (P_0 + \sum_{j=1}^m q_j) P M_k(C(Y)) P (P_0 + \sum_{j=1}^m q_j)$  such that

$$\|u_1^*(\psi'(f) \oplus \sum_{j=1}^m f(x_j) q_j) u_1 - \phi'(f) \oplus \sum_{j=1}^m f(x_j) q_j\| < \epsilon/4 \text{ for all } f \in \mathcal{F}. \quad (\text{e 3.49})$$

Define  $u = (u_1 \oplus P - (P_0 \oplus \sum_{j=1}^m q_j)) v \in P M_k(C(Y)) P$ . Then, by (e 3.49), (e 3.38), (e 3.35) and (e 3.36),

$$\|\text{ad } u \circ \psi(f) - \phi(f)\| < \epsilon \text{ for all } f \in \mathcal{F}. \quad (\text{e 3.50})$$

□

**Remark 3.3.** This statement could also be used in the proof of [11] to simplify some steps.

The statement and the proof of the above theorem could be simplified a slightly if  $\dim Y \leq 1$  because of the following version of Theorem 3.2 of [16].

**Theorem 3.4.** *Let  $X$  be a compact metric space and  $L : U(M_\infty(A)) \rightarrow R_+$  be a map. For any  $\epsilon > 0$  and any finite subset  $\mathcal{F} \subset C(X)$ , there exists a positive number  $\delta > 0$ , a finite subset  $\mathcal{G}$ , a finite subset  $\mathcal{P} \subset \underline{K}(C(X))$ , a finite subset  $\mathcal{U} \subset U(M_\infty(A))$ , an integer  $l \geq 1$  and  $\epsilon_1 > 0$  satisfying the following: if  $\phi, \psi : C(X) \rightarrow B$ , where  $B = \oplus_{j=1}^m C(X_j, M_{r(j)})$ ,  $X_j = [0, 1]$ , or  $X_j$  is a point, are two unital  $\delta$ - $\mathcal{G}$ -multiplicative contractive completely positive linear maps with*

$$[\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}} \text{ and } \text{cel}(\phi(v)^*\psi(v)) \leq L(u) \quad (\text{e 3.51})$$

*for all  $v \in \mathcal{U}$ , then there exists a unitary  $u \in M_{lm+1}(B)$  such that*

$$\|u^* \text{diag}(\phi(f), \sigma(f))u - \text{diag}(\psi(f), \sigma(f))\| < \epsilon \quad (\text{e 3.52})$$

*for all  $f \in \mathcal{F}$ , where  $\sigma(f) = \sum_{i=1}^m f(x_i)e_i$  for any  $\epsilon_1$ -dense set  $\{x_1, x_2, \dots, x_m\}$  and any set of mutually orthogonal projections  $\{e_1, e_2, \dots, e_m\}$  in  $M_{lm}(B)$  such that  $e_i$  is equivalent to  $\text{id}_{M_l(B)}$ .*

To prove the above theorem, we note that  $B$  has stable rank one,  $K_0$ -divisible rank  $T(n, k) = [n/k] + 1$ , exponential length divisible rank  $E(L, n) = 8\pi + L/n$  (see also Remark 1.1 of [16]). Therefore we have the following:

**Corollary 3.5.** *Let  $X$  be a simplicial finite CW complex, let  $\mathcal{F} \subset C(X)$  be a finite subset and let  $\epsilon > 0$ . There exists  $\eta_1 > 0$  with the following property .*

*For any  $\sigma_1 > 0$  and  $\sigma > 0$ , there exists  $\eta > 0$  and an integer  $K$  (which are independent of  $\sigma$ ), there exists  $\delta > 0$ , a finite subset  $\mathcal{G} \subset C(X)$ , a finite subset  $\mathcal{P} \subset \underline{K}(C(X))$ , a finite subset  $\mathcal{U} \subset U(M_\infty(C(X)))$  and a positive integer  $L$  satisfying the following:*

*Suppose that  $\phi, \psi : C(X) \rightarrow B = \oplus_{j=1}^m C(X_j, M_{r(j)})$ , where  $X_j = [0, 1]$ , or  $X_j$  is a point, are two unital homomorphisms such that*

$$\mu_{\tau \circ \phi}(O_{\eta_1}) \geq \sigma_1 \eta_1 \text{ and } \mu_{\tau \circ \phi}(O_{\eta}) \geq \sigma \eta \quad (\text{e 3.53})$$

$$|\tau \circ \phi(g) - \tau \circ \psi(g)| < \delta \text{ for all } g \in \mathcal{G} \quad (\text{e 3.54})$$

*and for all  $\tau \in T(B)$ ,*

$$\min_j \{\text{rank}(r(j))\} \geq L, \quad [\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}} \text{ and} \quad (\text{e 3.55})$$

$$\text{dist}(\phi^\dagger(z), \psi^\dagger(z)) < 1/8K\pi \quad (\text{e 3.56})$$

*for all  $z \in \mathcal{U}$ . Then there exists a unitary  $u \in B$  such that*

$$\|\phi(f) - \text{ad } u \circ \psi(f)\| < \epsilon \text{ for all } f \in \mathcal{F}. \quad (\text{e 3.57})$$

## 4 Almost multiplicative maps in finite dimensional $C^*$ -algebras

To begin, we would like to remind the reader that there exists a sequence of unital completely positive linear maps  $\phi_n : C(\mathbb{T} \times \mathbb{T}) \rightarrow M_n$  such that

$$\lim_{n \rightarrow \infty} \|\phi_n(f)\phi_n(g) - \phi_n(fg)\| = 0 \text{ for all } f, g \in C(\mathbb{T} \times \mathbb{T}) \quad (\text{e 4.58})$$

and  $\{\phi_n\}$  is away from homomorphisms (this was first discovered by D. Voiculescu [48]). Therefore  $\{\phi_n\}$  are not approximately unitarily equivalent to homomorphisms. This is because  $[\phi_n](b) \neq 0$  where  $b$  is the bott element. However, even when  $X$  is contractive, as long as  $\dim X > 2$ , one always has a sequence of contractive completely positive linear maps  $\phi_n : C(X) \rightarrow M_n$  such that (e 4.58) holds and  $\{\phi_n\}$  is away from any homomorphisms (see Theorem 4.2 of [15]). Therefore the condition on  $KK$ -theory (e 4.72) as well as the condition on the measure (e 4.74) in 4.3 are essential.

The following is a version of Theorem 4.6 of [27] and follows from that immediately.

**Lemma 4.1.** *Let  $X$  be a compact metric space, let  $\epsilon > 0$  and  $\mathcal{F} \subset C(X)$  be a finite subset. There exists  $\eta > 0$  which depends on  $\epsilon$  and  $\mathcal{F}$  for which*

$$|f(x) - f(x')| < \epsilon/8 \text{ for all } f \in \mathcal{F},$$

*if  $\text{dist}(x, x') < \eta$ , and for which the following holds:*

*For any  $\eta/2$ -dense subset  $\{x_1, x_2, \dots, x_m\}$  and any integer  $s \geq 1$  for which  $O_i \cap O_j = \emptyset$  ( $i \neq j$ ), where*

$$O_i = \{x \in X : \text{dist}(x_i, x) < \eta/2s\},$$

*and for any  $\sigma > 0$  for which  $1/2s > \sigma > 0$ , there exist  $\delta > 0$ , a finite subset  $\mathcal{G} \subset C(X)$  and a finite subset  $\mathcal{P} \subset \underline{K}(C(X))$  satisfying the following:*

*Suppose that  $\phi, \psi : C(X) \rightarrow A$  (for any unital simple  $C^*$ -algebra with tracial rank zero, infinite dimensional or finite dimensional) are two unital  $\delta$ - $\mathcal{G}$ -multiplicative contractive completely positive linear maps such that*

$$[\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}}, \quad (\text{e 4.59})$$

$$|\tau \circ \phi(g) - \tau \circ \psi(g)| < \delta \text{ for all } g \in \mathcal{G}, \tau \in T(A) \quad (\text{e 4.60})$$

$$\mu_{\tau \circ \phi}(O_i) \geq \sigma\eta \text{ and } \mu_{\tau \circ \psi}(O_i) \geq \sigma\eta \quad (\text{e 4.61})$$

*$i = 1, 2, \dots, m$ .*

*Then there exists a unitary  $u \in A$  such that*

$$\text{ad } u \circ \phi \approx_{\epsilon} \psi \text{ on } \mathcal{F}. \quad (\text{e 4.62})$$

**Lemma 4.2.** *Let  $X$  be a compact metric space, let  $\sigma_1 > 0$ ,  $1 > \eta_1 > 0$  and let  $\sigma > 0$ . For any  $\epsilon > 0$  and any finite subset  $\mathcal{F} \subset C(X)$ , there exist  $\eta > 0$  (which depends on  $\epsilon$  and  $\mathcal{F}$  but not  $\sigma_1$ ,  $\sigma$ , or  $\eta_1$ ),  $\delta > 0$ , a finite subset  $\mathcal{G}$  (both depend on  $\epsilon$ ,  $\mathcal{F}$ ,  $\sigma_1$ ,  $\sigma$  and  $\eta_1$ ) satisfying the following:*

*Suppose that  $\phi : C(X) \rightarrow M_n$  (for any integer  $n \geq 1$ ) is a  $\delta$ - $\mathcal{G}$ -multiplicative contractive completely positive linear map such that*

$$\mu_{\tau \circ \phi}(O_{\eta_1}) \geq \sigma_1\eta_1 \text{ and } \mu_{\tau \circ \phi}(O_{\eta}) \geq \sigma\eta \quad (\text{e 4.63})$$

*for all open balls with radius  $\eta_1$  and  $\eta$ , respectively.*

*Then there exists a unital homomorphism  $h : C(X) \rightarrow M_n$  such that*

$$|\tau \circ h(f) - \tau \circ \phi(f)| < \epsilon \text{ for all } f \in \mathcal{F} \quad (\text{e 4.64})$$

$$\mu_{\tau \circ h}(O_{\eta_1}) \geq (\sigma_1/2)\eta_1 \text{ and } \mu_{\tau \circ h}(O_{\eta}) \geq (\sigma/2)\eta \quad (\text{e 4.65})$$

*for all  $\tau \in T(A)$ .*

*Proof.* We apply Lemma 4.3 of [27]. Let  $\gamma > 0$  and  $\mathcal{F}_1 \subset C(X)$  be a finite subset.

It follows from Lemma 4.3 of [27] that, for a choice of  $\delta$  and  $\mathcal{G}$ , there is a projection  $p \in M_n$  and a unital homomorphism  $h_0 : C(X) \rightarrow pM_np$  such that

$$\|\phi(f) - [(1-p)\phi(f)(1-p) + h_0(f)]\| < \gamma \text{ for all } f \in \mathcal{F}_1 \quad (\text{e 4.66})$$

$$\text{and } \tau(1-p) < \gamma. \quad (\text{e 4.67})$$

Moreover, for any open ball  $O_{\eta}$  with radius  $\eta$ ,

$$\int_{O_{\eta}} h_0 d\mu_{\tau \circ h_0} > (\sigma/2)\eta \quad (\text{e 4.68})$$

(note  $\tau(p_k) > (\sigma\eta)/2$  in the statement of 4.3 of [27]).

Let  $h_1 : C(X) \rightarrow (1-p)M_n(1-p)$  be a unital homomorphism and define  $h = h_1 \oplus h_0$ . Therefore

$$\mu_{\tau \circ h}(O_\eta) > (\sigma/2)\eta \quad (\text{e 4.69})$$

for any open ball with radius  $\eta$ . Moreover,

$$|\tau \circ \phi(f) - \tau \circ h(f)| < 2\gamma \text{ for all } f \in \mathcal{F}_1. \quad (\text{e 4.70})$$

We choose  $\gamma < \epsilon/2$  and  $\mathcal{F}_1 \supset \mathcal{F}$ . It is easy to see that, if we choose sufficiently small  $\gamma$  and sufficiently large  $\mathcal{F}_1$ , we may also have

$$\mu_{\tau \circ h}(O_{\eta_1}) \geq (\sigma_1/2)\eta_1.$$

□

**Lemma 4.3.** *Let  $X$  be a path connected compact metric space, let  $\epsilon > 0$ ,  $\mathcal{F} \subset C(X)$  be a finite subset, let  $\sigma_1 > 0$ ,  $\sigma > 0$  and  $1 > \eta_1 > 0$ . Then, there exists  $\eta > 0$  (which depends on  $\epsilon$  and  $\mathcal{F}$  but not  $\sigma_1$ ,  $\sigma$  or  $\eta_1$ ),  $\delta > 0$ , a finite subset  $\mathcal{G} \subset C(X)$  and a finite subset  $\mathcal{P} \subset \underline{K}(C(X))$  satisfying the following:*

*Suppose that  $\phi : C(X) \rightarrow M_n$  (for any integer  $n \geq 1$ ) is a  $\delta$ - $\mathcal{G}$ -multiplicative contractive completely positive linear map such that*

$$\mu_{\tau \circ \phi}(O_\eta) \geq \sigma \cdot \eta \text{ and } \mu_{\tau \circ \phi}(O_{\eta_1}) \geq \sigma_1 \cdot \eta_1 \quad (\text{e 4.71})$$

*for all open balls with radius  $\eta$  and  $\eta_1$ , respectively, and*

$$[\phi]|_{\mathcal{P}} = [\pi_\xi]|_{\mathcal{P}} \quad (\text{e 4.72})$$

*for some point  $\xi \in X$ . Then there exists a unital homomorphism  $h : C(X) \rightarrow M_n$  such that*

$$\|\phi(f) - h(f)\| < \epsilon \text{ for all } f \in \mathcal{F}, \quad (\text{e 4.73})$$

$$\mu_{\tau \circ h}(O_{\eta_1}) \geq (\sigma_1/2)\eta_1 \text{ and } \mu_{\tau \circ h}(O_\eta) \geq (\sigma/2)\eta. \quad (\text{e 4.74})$$

*Proof.* Fix  $\epsilon > 0$ , a finite subset  $\mathcal{F} \subset C(X)$ ,  $\sigma_1$ ,  $\sigma$  and  $1 > \eta_1 > 0$ . Let  $\eta_2 > 0$  be a positive number such that

$$|f(x) - f(x')| < \epsilon/16,$$

if  $\text{dist}(x, x') < \eta_2$ . We may assume that  $\eta_2 < \eta_1$ . Let  $s$ ,  $\mathcal{G}_1$  (in place of  $\mathcal{G}$ ),  $\delta_1$  (in place of  $\delta$ ) and  $\mathcal{P} \subset \mathbf{P}(C(X))$  as in 4.1 (for the above  $\epsilon/2$ ,  $\eta_2$  and  $\sigma$ ).

Let  $\eta > 0$ ,  $\delta > 0$  and a finite subset  $\mathcal{G} \subset C(X)$  be as in Lemma 4.2 required for  $\gamma$  (in place of  $\epsilon$ ),  $\mathcal{G}_1 \cup \mathcal{F}$  (in place of  $\mathcal{F}$ ),  $\sigma$  (with  $\sigma_1 = \sigma$ ) and  $\eta_2$  (in place of  $\eta_1$ ) above.

Now suppose that  $\phi : C(X) \rightarrow M_n$  is a  $\delta$ - $\mathcal{G}$ -multiplicative contractive completely positive linear map satisfying the assumption with the above  $\eta$ ,  $\delta$ ,  $\mathcal{G}$  and  $\mathcal{P}$ . By applying 4.2, one obtains a unital homomorphism  $h_1 : C(X) \rightarrow M_n$  such that

$$|\tau \circ \phi(g) - \tau \circ h_1(g)| < \delta_1 \text{ for all } g \in \mathcal{G}_1 \quad (\text{e 4.75})$$

$$\mu_{\tau \circ h_1}(O_\eta) \geq (\sigma/2)\eta \text{ and} \quad (\text{e 4.76})$$

$$\mu_{\tau \circ h_1}(O_{\eta_2}) \geq (\sigma/2)\eta_2. \quad (\text{e 4.77})$$

Since  $X$  is a path connected,

$$[h_1] = [\pi_\xi].$$

It follows that

$$[h_1]|_{\mathcal{P}} = [\phi]|_{\mathcal{P}}.$$

It then follows from 4.1 that there exists a unitary  $u \in M_n$  such that

$$\text{ad } u \circ h \approx_{\epsilon} \phi \text{ on } \mathcal{F}.$$

Put  $h = \text{ad } u \circ h_1$ . One also has that

$$\mu_{\tau \circ h}(O_{\eta}) = \mu_{\tau \circ h_1}(O_{\eta}) \geq \sigma \cdot \eta/2.$$

Note that, if one choose  $\delta_1$  sufficiently smaller and  $\mathcal{G}_1$  sufficiently larger, one may also require that

$$\mu_{\tau \circ h}(O_{\eta_1}) \geq (\sigma_1/2)\eta_1.$$

□

## 5 The Basic Homotopy Lemma revisited

The purpose of this section is to present Lemma 5.4. It is slightly different from Theorem 7.4 of [29]. We will give a brief proof.

We begin with an easy fact:

**Lemma 5.1.** *Let  $X$  be a compact metric space and let  $A$  be a finite dimensional  $C^*$ -algebra. Suppose that  $\phi : C(X) \rightarrow A$  is a unital homomorphism and  $u \in A$  is a unitary such that*

$$\phi(f)u = u\phi(f) \text{ for all } f \in C(X).$$

*Then there exists a continuous path of unitaries  $\{u(t) : t \in [0, 1]\}$  such that*

$$u(0) = u, \quad u(1) = 1, \quad \phi(f)u(t) = u(t)\phi(f) \text{ for all } f \in C(X) \text{ and}$$

$$\text{Length}(\{u(t)\}) \leq \pi.$$

*Proof.* Define  $H : C(X \times \mathbb{T}) \rightarrow A$  by  $H(f \otimes g) = \phi(f)g(u)$  for  $f \in C(X)$  and  $g \in C(\mathbb{T})$ . Note that  $H(C(X \times \mathbb{T}))$  is a commutative finite dimensional  $C^*$ -algebra. The lemma follows immediately. □

**Lemma 5.2.** *Let  $X$  be a compact path connected metric space, let  $\epsilon > 0$  and let  $\mathcal{F} \subset C(X)$  be a finite subset. There exists  $\eta > 0$  such that the following holds:*

*For any  $\sigma > 0$ , there exists an integer  $s \geq 1$ ,  $\delta > 0$ , a finite subset  $\mathcal{G} \subset C(X)$  and a finite subset  $\mathcal{P} \subset \underline{K}(C(X))$  satisfying the following:*

*Suppose that  $\phi : C(X) \rightarrow M_n$  (for some integer  $n$ ) is unital homomorphism and a unitary  $u \in M_n$  such that there is  $\delta$ - $\mathcal{G}$ -multiplicative contractive completely positive linear map  $\Phi : C(X \times \mathbb{T}) \rightarrow M_n$  such that*

$$\|\Phi(f \otimes 1) - \phi(f)\| < \delta \text{ for all } f \in \mathcal{G}, \quad \|u - \Phi(1 \otimes z)\| < \delta, \quad (\text{e5.78})$$

*where  $z$  is the identity map on the unit circle,*

$$\text{Bott}(\phi, u)|_{\mathcal{P}} = \{0\} \text{ and} \quad (\text{e5.79})$$

$$\mu_{\tau \circ \Phi}(O_{\eta/2s}) \geq \sigma\eta \quad (\text{e5.80})$$

*for any open ball  $O_{\eta/2s}$  of  $X \times \mathbb{T}$  with radius  $\eta/2s$ .*

*Then there is a continuous path of unitaries  $\{u(t) : t \in [0, 1]\}$  such that*

$$u(0) = u, \quad u(1) = 1, \quad \|\phi(f), u(t)\| < \epsilon \text{ for all } f \in \mathcal{F} \text{ and}$$

$$\text{length}(\{u(t)\}) \leq \pi + \epsilon\pi.$$



*Proof.* Let  $\epsilon > 0$  and  $\mathcal{F}$  be as in the statement. We may assume that  $\epsilon < 1/4$ . Let  $Y = X \times \mathbb{T}$  and

$$\mathcal{F}_1 = \{f \times g : f \in \mathcal{F} \cup \{1\}, g = 1 \text{ and } g = z\},$$

where  $z$  is the identity map of the unit circle.

Let  $\eta > 0$  be as in Lemma 4.3 for  $\mathcal{F}_1$  (instead of  $\mathcal{F}$ ) and  $\epsilon/4$  (instead of  $\epsilon$ ) for  $Y$ . Fix  $\sigma_1 = \sigma > 0$  (and  $\eta_1 = \eta$ ). Let  $s \geq 1$ ,  $\delta_0$  (in place of  $\delta$ ),  $\mathcal{G}_1$  (in place of  $\mathcal{G}$ ) and  $\mathcal{Q} \subset \underline{K}(C(X \times \mathbb{T}))$  (in place of  $\mathcal{P}$ ) be as required by 4.3 for the above  $\epsilon/4$ ,  $\mathcal{F}$ ,  $\eta$  and  $\sigma_1$  (and for  $Y$ ). There is  $\delta_1 > 0$ , a finite subset  $\mathcal{G}_1 \subset C(X \times \mathbb{T})$  and a finite subset  $\mathcal{Q} \subset \beta(\underline{K}(C(X)))$  such that

$$[\Psi]|_{\beta(\mathcal{Q})} = [\pi_\xi]|_{\beta(\mathcal{Q})}$$

for any  $\delta_1$ - $\mathcal{G}_1$ -multiplicative contractive completely positive linear map for which

$$\|\Psi(f \otimes 1) - \phi(f)\| < \delta_1 \text{ for all } f \in \mathcal{G}_1, \quad \|\Phi(1 \otimes z) - v\| < \delta_1 \quad (\text{e 5.81})$$

$$\text{and } \text{Bott}(\phi, v)|_{\mathcal{P}} = \{0\}, \quad (\text{e 5.82})$$

(for any unitary  $v \in M_n$  satisfying the above).

Now suppose that  $\phi$  and  $u$  satisfy the assumption for the above  $\eta$ ,  $\delta$ ,  $\mathcal{G}$  and  $\mathcal{P}$ . It follows from 4.3 that there is a unital homomorphism  $H : C(X \times \mathbb{T}) \rightarrow M_n$  such that

$$\|\Phi(g) - H(g)\| < \epsilon/4 \text{ for all } g \in \mathcal{F}_1. \quad (\text{e 5.83})$$

It follows from 5.1 that there exist a continuous path of unitaries  $\{u(t) : t \in [1/4, 1]\}$  such that

$$u(1/4) = H(1 \otimes z), \quad u(1) = 1, \quad (\text{e 5.84})$$

$$u(t)H(g \otimes 1) = H(g \otimes 1)u(t) \text{ for all } g \in C(X), \quad t \in \text{ and } \quad (\text{e 5.85})$$

$$\text{Length}(\{u(t) : t \in [1/4, 1]\}) \leq \pi. \quad (\text{e 5.86})$$

Since

$$\|u - H(1 \otimes z)\| < \epsilon/2,$$

There is a continuous path of unitaries  $\{u(t) : t \in [0, 1/4]\}$  such that

$$u(0) = u, \quad u(1/4) = H(1 \otimes z) \text{ and } \text{Length}(\{u(t) : t \in [0, 1/4]\}) \leq \epsilon \cdot \pi.$$

The lemma then follows. □

**Lemma 5.3.** *Let  $X$  be a compact metric space without isolated points,  $\epsilon > 0$  and  $1 \in \mathcal{F} \subset C(X)$  be a finite subset. Let  $l$  be a positive integer for which  $256\pi M/l < \epsilon$ , where  $M = \max\{1, \max\{\|f\| : f \in \mathcal{F}\}\}$ . Then, there exists  $\eta > 0$  (which depends on  $\epsilon$  and  $\mathcal{F}$ ) for any finite  $\eta/2$ -dense subset  $\{x_1, x_2, \dots, x_N\}$  of  $X$  for which  $O_i \cap O_j = \emptyset$  ( $i \neq j$ ), where*

$$O_i = \{x \in X : \text{dist}(x, x_i) < \eta/2s\}$$

*for some integer  $s \geq 1$  and for any  $\sigma > 0$  for which  $\sigma < 1/2s$ , and for any  $\delta_0 > 0$  and any finite subset  $\mathcal{G}_0 \subset C(X \otimes \mathbb{T})$ , there exists a finite subset  $\mathcal{G} \subset C(X)$  and there exists  $\delta > 0$  satisfying the following:*

Suppose that  $A$  is a unital separable simple  $C^*$ -algebra with tracial rank zero (infinite dimensional or finite dimensional),  $h : C(X) \rightarrow A$  is a unital homomorphism and  $u \in A$  is a unitary such that

$$\|[h(a), u]\| < \delta \text{ for all } a \in \mathcal{G} \text{ and } \mu_{\tau \circ h}(O_i) \geq \sigma\eta \text{ for all } \tau \in T(A). \quad (\text{e5.87})$$

Then there is a  $\delta_0$ - $\mathcal{G}_0$ -multiplicative contractive completely positive linear map  $\phi : C(X) \otimes C(\mathbb{T}) \rightarrow A$  and a rectifiable continuous path  $\{u_t : t \in [0, 1]\}$  such that

$$u_0 = u, \quad \|\phi(a \otimes 1), u_t\| < \epsilon \text{ for all } a \in \mathcal{F}, \quad (\text{e5.88})$$

$$\|\phi(a \otimes 1) - h(a)\| < \epsilon, \quad \|\phi(a \otimes z) - h(a)u\| < \epsilon \text{ for all } a \in \mathcal{F}, \quad (\text{e5.89})$$

where  $z \in C(\mathbb{T})$  is the standard unitary generator of  $C(\mathbb{T})$ , and

$$\mu_{\tau \circ \phi}(O(x_i \times t_j)) > \frac{\sigma_1}{2l}\eta, \quad i = 1, 2, \dots, m, j = 1, 2, \dots, l \quad (\text{e5.90})$$

for all  $\tau \in T(A)$ , where  $t_1, t_2, \dots, t_l$  are  $l$  points on the unit circle which divide  $\mathbb{T}$  into  $l$  arcs evenly and where

$$O(x_i \times t_j) = \{x \times t \in X \times \mathbb{T} : \text{dist}(x, x_i) < \eta/2s \text{ and } \text{dist}(t, t_j) < \pi/4sl\} \text{ for all } \tau \in T(A)$$

(so that  $O(x_i \times t_j) \cap O(x_{i'} \times t_{j'}) = \emptyset$  if  $(i, j) \neq (i', j')$ ). Moreover,

$$\text{Length}(\{u_t\}) \leq \pi + \epsilon\pi. \quad (\text{e5.91})$$

*Proof.* The only difference of this lemma and Lemma 6.4 of [29] is that in the statement of Lemma 6.4 of [29]  $h$  is assumed to be monomorphism. However, for the case that  $A$  is infinite dimensional, it is the condition that

$$\mu_{\tau \circ h}(O_i) \geq \sigma \cdot \eta$$

for all  $\tau \in T(A)$  which is actually used. The existence of a monomorphism  $h$  implies that  $A$  is infinite dimensional.

In the case that  $M_n, pAp$  may not have enough projections, a modification is needed for the case that  $A = M_n$  for some integer  $n$ . Let  $\eta > 0$  be such that

$$|f(x) - f(x')| < \epsilon/32 \text{ for all } f \in \mathcal{F},$$

if  $\text{dist}(x, x') < \eta$ . Suppose that  $y_1, y_2, \dots, y_m \in X$  and  $s_1 \geq 1$  such that

$$G_i \cap G_j = \emptyset \text{ if } i \neq j,$$

where  $G_i = B_{\eta_1/2s_1}(y_i)$ ,  $i = 1, 2, \dots, m$ . Let  $\{x_1, x_2, \dots, x_{2ml}\}$  be another subset of  $X$  such that each  $G_i$  contains  $2l$  many points.

Now let  $\delta_0$  and  $\mathcal{G}_0$  be given. Then there is  $s > s_1$  such that

$$O_i \cap O_j = \emptyset, \text{ if } i \neq j,$$

where  $O_j = B_{\eta_1/2s}(x_j)$ ,  $j = 1, 2, \dots, m + 2l$ . Let  $0 < \sigma < 1/2s$ . Let  $\sigma_1 = 2l\sigma$ . Let  $\delta$  and  $\mathcal{G}$  be required by Lemma 6.4 of [29] for the above  $\epsilon, \mathcal{F}, l, \eta, s, \sigma_1, \delta_0$  and  $\mathcal{G}_0$ .

Now suppose that  $h : C(X) \rightarrow A$  is a unital homomorphism and  $u \in A$  is a unitary such that

$$\|[h(f), u]\| < \delta \text{ for all } f \in \mathcal{G} \text{ and } \mu_{\tau \circ h}(O_i) \geq \sigma\eta.$$

Then

$$\mu_{\tau \circ h}(G_i) \geq \sigma_1 \eta \geq 2l\sigma\eta.$$

In particular  $pAp$  contains  $2l - 1$  mutually orthogonal and mutually equivalent non-zero projections. Thus the proof of Lemma 6.4 of [29] applies.  $\square$

**Lemma 5.4.** *Let  $X$  be a finite CW complex,  $\mathcal{F} \subset C(X)$  be a finite subset and  $\epsilon > 0$  be a positive number. Let  $\sigma > 0$ . There exists  $\eta > 0$  (which depends on  $\epsilon$  and  $\mathcal{F}$  but not on  $\sigma$ ),  $\delta > 0$ , a finite subset  $\mathcal{G} \subset C(X)$  and a finite subset  $\mathcal{P} \subset \underline{K}(C(X))$  satisfying the following:*

*Suppose that  $\phi : C(X) \rightarrow A$ , where  $A$  is a unital separable simple  $C^*$ -algebra with tracial rank zero (infinite or finite dimensional), is a unital homomorphism with*

$$\mu_{\tau \circ \phi}(O_{\eta/2}) \geq \sigma\eta \tag{e 5.92}$$

*for any open ball with radius  $\eta/2$  and a unitary  $u \in A$  such that*

$$\|[\phi(g), u]\| < \delta \text{ for all } g \in \mathcal{G} \text{ and } \text{Bott}(\phi, u)|_{\mathcal{P}} = \{0\}. \tag{e 5.93}$$

*Then there exists a continuous path of unitaries  $\{u_t : t \in [0, 1]\}$  such that*

$$u_0 = u, \quad u_1 = 1, \quad \|[\phi(f), u_t]\| < \epsilon \tag{e 5.94}$$

*for all  $f \in \mathcal{F}$  and  $t \in [0, 1]$  and*

$$\text{length}(\{u_t\}) \leq 2\pi + \epsilon.$$

*Proof.* For the case that  $A$  is infinite dimensional, the proof is exactly the same of that Theorem 7.4 of [29]. The proof is slightly different from that of Theorem 7.4 of [29] when  $A$  is finite dimensional. However, it is a combination of 5.3 and 5.2 just as the proof of Theorem 7.4 of [29].  $\square$

**Remark 5.5.** Consider the case that  $A$  is finite dimensional. Let  $X$  be a finite CW complex. Suppose that  $\psi : C(X \times \mathbb{T}) \rightarrow A$  is a  $\delta$ - $\mathcal{G}$ -multiplicative contractive completely positive linear map. Since  $K_1(M_n) = \{0\}$ , it is clear that, with sufficiently small  $\delta > 0$  and sufficiently large finite subset  $\mathcal{G}$ ,  $[\psi]|_{\beta^{(0)}(K_0(C(X)))} = \{0\}$ . It follows that

$$\text{bott}_0(\phi, u) = \{0\},$$

if  $\|[\phi(f), u]\| < \delta$  for any unitary and all  $f \in \mathcal{F}$  for a sufficiently large finite subset  $\mathcal{F} \subset C(X)$  (and sufficiently small  $\delta$ ).

Fix an integer  $k$ . With sufficiently large  $\mathcal{G}$  and sufficiently small  $\delta$ , it is clear that  $[\psi]|_{\beta(K_0(C(X)/k\mathbb{Z}))} = \{0\}$ . Suppose that  $K_0(C(X))$  is torsion free and  $[\psi]|_{\beta(K_1(C(X)))} = 0$ . It is then easy to check that

$$[\psi]|_{\beta(K_1(C(X)/k\mathbb{Z}))} = 0,$$

provided that  $\delta$  is sufficiently small and  $\mathcal{G}$  is sufficiently large.

From this, in the statement of Theorem 5.4, it suffices to replace  $\underline{K}(C(X))$  by  $K_1(C(X))$  and to replace (e 5.79) by

$$\text{bott}_1(\phi, u)|_{\mathcal{P}} = 0,$$

provided that  $K_0(C(X))$  is torsion free.

## 6 Homotopy and unitary equivalence

Let  $X$  be a locally path connected compact metric space. Let  $\phi, \psi : C(X) \rightarrow A$  be two unital homomorphisms, where  $A$  is a finite dimensional  $C^*$ -subalgebra. In this section, we will show that  $\phi$  and  $\psi$ , up to some homotopy, are unitary equivalent if they are close and they induce similar measure. See 6.2 below.

**Lemma 6.1.** *Let  $X$  be a connected compact metric space. For any  $\eta > 0$  and  $\sigma > 0$ , there is  $\delta = (\sigma\eta/16)$  and there is a finite subset  $\mathcal{G} \subset C(X)$  such that if  $\phi, \psi : C(X) \rightarrow A$  are two unital homomorphisms, where  $A$  is a unital  $C^*$ -algebra with a tracial state  $\tau$ , such that*

$$|\tau \circ \phi(g) - \tau \circ \psi(g)| < \delta \text{ for all } g \in \mathcal{G} \quad (\text{e 6.95})$$

$$\mu_{\tau \circ \phi}(O_{\eta/8}) \geq \sigma\eta/8 \quad \text{and} \quad \mu_{\tau \circ \psi}(O_{\eta/8}) \geq \sigma\eta/8, \quad (\text{e 6.96})$$

then, for any compact subset  $F \subset X$ ,

$$\mu_{\tau \circ \phi}(F) \leq \mu_{\tau \circ \psi}(B_\eta(F)) \quad \text{and} \quad \mu_{\tau \circ \psi}(F) \leq \mu_{\tau \circ \phi}(B_\eta(F)), \quad (\text{e 6.97})$$

where

$$B_\eta(F) = \{x \in X : \text{dist}(x, F) < \eta\}.$$

*Proof.* There are finitely many open balls  $B_{\eta/8}(x_1), B_{\eta/8}(x_2), \dots, B_{\eta/8}(x_N)$  with radius  $\eta/8$  covers  $X$ . It is an easy exercise to show that there is a finite subset  $\mathcal{G}$  of  $C(X)$  satisfying the following: if (e 6.95) holds, then, for any subset  $S$  of  $\{1, 2, \dots, N\}$ ,

$$\mu_{\tau \circ \phi}(\cup_{i \in S} B_{\eta/8}(x_i)) \leq \mu_{\tau \circ \psi}(\cup_{i \in S} B_{\eta/4}(x_i)) + \delta \quad \text{and} \quad (\text{e 6.98})$$

$$\mu_{\tau \circ \psi}(\cup_{i \in S} B_{\eta/8}(x_i)) \leq \mu_{\tau \circ \phi}(\cup_{i \in S} B_{\eta/4}(x_i)) + \delta. \quad (\text{e 6.99})$$

If  $\overline{\cup_{i \in S} B_{3\eta/4}(x_i)} = X$ , then

$$\mu_{\tau \circ \phi}(\cup_{i \in S} B_{\eta/8}(x_i)) \leq \mu_{\tau \circ \psi}(\overline{\cup_{i \in S} B_{3\eta/4}(x_i)}) \quad \text{and} \quad (\text{e 6.100})$$

$$\mu_{\tau \circ \psi}(\cup_{i \in S} B_{\eta/8}(x_i)) \leq \mu_{\tau \circ \phi}(\overline{\cup_{i \in S} B_{3\eta/4}(x_i)}). \quad (\text{e 6.101})$$

Otherwise, since  $X$  is path connected, there is an open ball  $O$  of  $X$  with radius  $\eta/8$  such that

$$O \cap (\cup_{i \in S} B_{\eta/4}(x_i)) = \emptyset \quad \text{and} \quad O \subset \cup_{i \in S} B_\eta(x_i).$$

Thus, by (e 6.98), (e 6.100) and (e 6.96),

$$\mu_{\tau \circ \phi}(\cup_{i \in S} B_{\eta/8}(x_i)) \leq \mu_{\tau \circ \psi}(\cup_{i \in S} B_\eta(x_i)). \quad (\text{e 6.102})$$

Now for any compact subset  $F$ , there is  $S \subset \{1, 2, \dots, N\}$  such that

$$F \subset \cup_{i \in S} B_{\eta/8}(x_i) \quad \text{and} \quad F \cap B_{\eta/8}(x_i) \neq \emptyset \text{ for all } i \in S. \quad (\text{e 6.103})$$

It follows that

$$\mu_{\tau \circ \phi}(F) \leq \mu_{\tau \circ \phi}(\cup_{i \in S} B_{\eta/8}(x_i)) \quad (\text{e 6.104})$$

$$\leq \mu_{\tau \circ \psi}(\cup_{i \in S} B_\eta(x_i)) \leq \mu_{\tau \circ \psi}(B_\eta(F)). \quad (\text{e 6.105})$$

Exactly the same argument shows that the other inequality of (e 6.97) also holds.  $\square$

**Lemma 6.2.** *Let  $X$  be a locally path connected compact metric space without isolated points, let  $\epsilon > 0$  and let  $\mathcal{F} \subset C(X)$  be a finite subset. Let  $\eta > 0$  be such that*

$$|f(x) - f(x')| < \epsilon/2 \text{ for all } f \in \mathcal{F},$$

*provided that  $\text{dist}(x, x') < \eta$  and such that any open ball  $B_\eta$  with radius  $\eta$  is path connected.*

*Let  $\sigma > 0$ . There is  $\delta > 0$  and there exists a finite subset  $\mathcal{G} \subset C(X)$  satisfying the following: For any two unital homomorphisms  $\phi, \psi : C(X) \rightarrow M_n$  (for any  $n \geq 1$ ) for which*

$$\|\phi(f) - \psi(f)\| < \delta \text{ for all } f \in \mathcal{G} \text{ and} \quad (\text{e 6.106})$$

$$\mu_{\tau \circ \phi}(O_{\eta/24}), \mu_{\tau \circ \psi}(O_{\eta/24}) \geq \sigma\eta \quad (\text{e 6.107})$$

*for any open balls with radius  $\eta/24$ , there exist two unital homomorphisms  $\Phi_1, \Phi_2 : C([0, 1], M_n)$  such that*

$$\pi_0 \circ \Phi_1 = \phi, \quad \pi_0 \circ \Phi_2 = \psi, \quad (\text{e 6.108})$$

$$\|\pi_t \circ \Phi_1(f) - \phi(f)\| < \epsilon, \quad \|\pi_t \circ \Phi_2(f) - \psi(f)\| < \epsilon \quad (\text{e 6.109})$$

*for all  $f \in \mathcal{F}$  and  $t \in [0, 1]$ , and there is a unitary  $u \in M_n$  such that*

$$\text{ad } u \circ \pi_1 \circ \Phi_1 = \pi_1 \circ \Phi_2. \quad (\text{e 6.110})$$

*Proof.*  $X$  is a union of finitely many connected and locally path connected compact metric spaces. It is clear that the general case can be reduced to the case that  $X$  is a connected and locally path connected compact metric space.

We will apply the so-called Marriage Lemma (see [18]). Let  $\delta$  and  $\mathcal{G}$  be in 6.1 corresponding to  $\eta/3$  and  $\sigma$ . We may assume that  $\mathcal{G} \supset \mathcal{F}$ . We may write that

$$\phi(f) = \sum_{i=1}^{N_1} f(x_i) p_i \text{ and } \psi(f) = \sum_{j=1}^{N_2} f(y_j) q_j \quad (\text{e 6.111})$$

for all  $f \in C(X)$ , where  $\{p_1, p_2, \dots, p_{N_1}\}$  and  $\{q_1, q_2, \dots, q_{N_2}\}$  are two sets of mutually orthogonal projections such that  $\sum_{i=1}^{N_1} p_i = 1 = \sum_{j=1}^{N_2} q_j$ .

By 6.1,

$$\mu_{\tau \circ \phi}(F) < \mu_{\tau \circ \psi}(B_{\eta/3}(F)) \text{ and } \mu_{\tau \circ \psi}(F) < \mu_{\tau \circ \phi}(B_{\eta/3}(F)) \quad (\text{e 6.112})$$

for any compact subset  $F \subset X$ .

Suppose that  $p_i$  has rank  $r(i)$ . Choose  $r(i)$  many points  $\{x_{i,1}, x_{i,2}, \dots, x_{i,r(i)}\} \subset B_{\eta/3}(x_i)$  and define

$$\phi_1(f) = \sum_{i=1}^{N_1} \left( \sum_{k=1}^{r(i)} f(x_{i,k}) e_{i,k} \right) \text{ for all } f \in C(X),$$

where  $\{e_{i,1}, e_{i,2}, \dots, e_{i,r(i)}\}$  is a set of mutually orthogonal rank one projections such that  $\sum_{k=1}^{r(i)} e_{i,k} = p_i$ . It follow that

$$\mu_{\tau \circ \phi_1}(F) \leq \mu_{\tau \circ \phi}(B_{\eta/3}(F)) \text{ and } \mu_{\tau \circ \phi}(F) \leq \mu_{\tau \circ \phi_1}(B_{\eta/3}(F)) \quad (\text{e 6.113})$$

for any compact subset  $F \subset X$ . Since  $B_\eta(x)$  is path connected for every  $x \in X$ , there is a unital homomorphism  $\Phi_1 : C(X) \rightarrow C([0, 1], M_n)$  such that

$$\pi_0 \circ \Phi_1 = \phi, \quad \pi_1 \circ \Phi_1 = \phi_1 \text{ and} \quad (\text{e 6.114})$$

$$\|\pi_t \circ \Phi_1(f) - \phi(f)\| < \epsilon/2 \text{ for all } f \in \mathcal{F} \text{ and } t \in [0, 1]. \quad (\text{e 6.115})$$

We rewrite

$$\phi_1(f) = \sum_{i=1}^n f(x'_i) e_i \text{ for all } f \in C(X), \quad (\text{e 6.116})$$

where each  $e_i$  is a rank one projection and  $x'_i$  is a point in  $X$ ,  $i = 1, 2, \dots, n$ , and  $\sum_{i=1}^n e_i = 1$ . Similarly, there is a unital homomorphism  $\Phi'_2 : C(X) \rightarrow C([0, 1/2], M_n)$  such that

$$\pi_0 \circ \Phi'_2 = \psi, \quad \pi_{1/2} \circ \Phi'_2 = \psi_1 \text{ and} \quad (\text{e 6.117})$$

$$\|\pi_t \circ \Phi'_2(f) - \psi(f)\| < \epsilon/2 \text{ for all } f \in \mathcal{F} \text{ and } t \in [0, 1/2], \quad (\text{e 6.118})$$

where

$$\psi_1(f) = \sum_{i=1}^n f(y'_i) e'_i \text{ for all } f \in C(X), \quad (\text{e 6.119})$$

where each  $e'_i$  is a rank projection,  $y'_i$  is a point in  $X$ ,  $i = 1, 2, \dots, n$  and  $\sum_{i=1}^n e'_i = 1$ . Moreover,

$$\mu_{\tau \circ \psi_1}(F) \leq \mu_{\tau \circ \psi}(B_{\eta/3}(F)) \text{ and } \mu_{\tau \circ \psi}(F) \leq \mu_{\tau \circ \psi_1}(B_{\eta/3}(F)) \quad (\text{e 6.120})$$

for any compact subset  $F \subset X$ . Combining (e 6.112), (e 6.113) and (e 6.120), one has

$$\mu_{\tau \circ \phi_1}(F) \leq \mu_{\tau \circ \phi}(B_{\eta/3}(F)) < \mu_{\tau \circ \psi}(B_{2\eta/3}(F)) \quad (\text{e 6.121})$$

$$\leq \mu_{\tau \circ \psi_1}(B_{\eta}(F)) \text{ and} \quad (\text{e 6.122})$$

$$\mu_{\tau \circ \psi_1}(F) < \mu_{\tau \circ \phi_1}(B_{\eta}(F)) \quad (\text{e 6.123})$$

for any compact subset  $F \subset X$ .

By the Marriage Lemma (see [18]), there is a permutation  $\Delta : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$  such that

$$\text{dist}(x'_i, y'_{\Delta(i)}) < \eta, \quad i = 1, 2, \dots, n. \quad (\text{e 6.124})$$

Define  $\psi_2 : C(X) \rightarrow M_n$  by

$$\psi_2(f) = \sum_{i=1}^n f(x'_i) e'_{\Delta(i)} \text{ for all } f \in C(X). \quad (\text{e 6.125})$$

Since every open ball of radius  $\eta$  is path connected, one obtains another unital homomorphism  $\Phi''_2 : C(X) \rightarrow C([1/2, 1], M_n)$

$$\pi_1 \circ \Phi''_2 = \psi_2, \quad \pi_{1/2} \circ \Phi''_2 = \psi_1 \text{ and} \quad (\text{e 6.126})$$

$$\|\pi_t \circ \Phi''_2(f) - \psi_1(f)\| < \epsilon/2 \text{ for all } f \in \mathcal{F}. \quad (\text{e 6.127})$$

Now define  $\Phi_2 : C([0, 1], M_n)$  by  $\pi_t \circ \Phi_2 = \pi_t \circ \Phi'_2$  for  $t \in [0, 1/2]$  and  $\pi_t \circ \Phi_2 = \pi_t \circ \Phi''_2$  for  $t \in [1/2, 1]$ . Then  $\Phi_1$  and  $\Phi_2$  satisfy (e 6.108) and (e 6.109). Moreover, by (e 6.116) and (e 6.125), there exists a unitary  $u \in M_n$  such that

$$\text{ad } u \circ \phi_1 = \psi_2 = \pi_1 \circ \Phi_2.$$

□

**Remark 6.3.** If  $\phi(f) = \sum_{i=1}^{N_1} f(x_i) p_i$  as in the proof, let  $C_0$  be the finite dimensional commutative  $C^*$ -subalgebra generated by mutually orthogonal projections  $\{p_1, p_2, \dots, p_{N_1}\}$ . Then, we actually proved that there is a finite dimensional commutative  $C^*$ -subalgebra  $C_1 \supset C_0$  with  $1_{C_1} = 1_{C_0}$  such that  $\pi_t \circ \Phi(C(X)) \subset C_1$  for all  $t \in [0, 1]$ . Similarly, if  $\psi(f) = \sum_{j=1}^{N_2} f(y_j) q_j$ , let  $C'_0$  be the finite dimensional commutative  $C^*$ -subalgebra generated by mutually orthogonal projections  $\{q_1, q_2, \dots, q_{N_2}\}$ . Then, as in the proof, there is a finite dimensional commutative  $C^*$ -subalgebra  $C'_1 \supset C'_0$  with  $1_{C'_1} = 1_{C'_0}$  such that  $\pi_t \circ \Psi(C(X)) \subset C'_1$  for all  $t \in [0, 1]$ .

## 7 Local homotopy lemmas

**Lemma 7.1.** *Let  $X$  be a finite CW complex with torsion  $K_1(C(X))$  and torsion free  $K_0(C(X))$ . Let  $\epsilon > 0$ ,  $\mathcal{F} \subset C(X)$  be a finite subset and let  $\sigma > 0$ . There exist  $\eta > 0$  (which depends on  $\epsilon$  and  $\mathcal{F}$  but not  $\sigma$ ), a finite subset  $\mathcal{G} \subset C(X)$  and  $\delta > 0$  satisfying the following:*

*Suppose that  $\phi, \psi : C(X) \rightarrow M_n$  (for any integer  $n$ ) are two unital homomorphisms such that*

$$\|\phi(f) - \psi(f)\| < \delta \text{ for all } f \in \mathcal{G} \quad (\text{e 7.128})$$

$$\mu_{\tau \circ \phi}(O_\eta) \geq \sigma\eta \text{ and } \mu_{\tau \circ \psi}(O_\eta) \geq \sigma\eta \quad (\text{e 7.129})$$

*for any open ball  $O_\eta$  of radius  $\eta$ , where  $\tau$  is the normalized trace on  $M_n$ , and*

$$\text{ad } u \circ \phi = \psi \quad (\text{e 7.130})$$

*for some unitary  $u \in A$ . Then, there exists a homomorphism  $\Phi : C(X) \rightarrow C([0, 1], M_n)$  such that*

$$\pi_0 \circ \Phi = \phi, \quad \pi_1 \circ \Phi = \psi \text{ and}$$

$$\|\psi(f) - \pi_t \circ \Phi(f)\| < \epsilon \text{ for all } f \in \mathcal{F}.$$

*Proof.* It is easy to see that the general case can be reduced to the case that  $X$  is connected.

Let  $\epsilon > 0$ ,  $\mathcal{F} \subset C(X)$  be a finite subset and let  $\sigma > 0$ . Let  $\eta_1 > 0$  (in place of  $\eta$ ),  $\delta > 0$  and a finite subset  $\mathcal{G} \subset C(X)$  and a finite subset  $\mathcal{P} \subset \underline{K}(C(X))$  be required by 5.4 for  $\epsilon/2$ ,  $\mathcal{F}$  and  $\sigma/2$ . Let  $\eta = \eta_1/2$ .

By 5.5, we may assume that  $\mathcal{P} \subset K_1(C(X))$ . Since  $K_1(C(X))$  is torsion and  $K_0(M_n)$  is free, for sufficiently small  $\delta$  and sufficiently large  $\mathcal{G}$ , for any pair of  $\phi$  and  $u$  for which  $\|\phi(g), u\| < \delta$  for all  $g \in \mathcal{G}$ ,

$$\text{bott}_1(\phi, u)|_{\mathcal{P}} = 0.$$

We may assume that  $\delta$  and  $\mathcal{G}$  have this property. We may further assume that  $\delta < \epsilon/2$  and  $\mathcal{F} \subset \mathcal{G}$ .

Now we assume that  $\phi, \psi$  and  $u$  satisfy the assumption of the lemma for the above  $\eta, \delta$  and  $\mathcal{G}$ . Then

$$\mu_{\tau \circ \phi}(O_{\eta_1/2}) \geq \sigma\eta_1/2 = (\sigma/2)\eta_1 \text{ and} \quad (\text{e 7.131})$$

$$\mu_{\tau \circ \psi}(O_{\eta_1/2}) \geq (\sigma/2)\eta_1. \quad (\text{e 7.132})$$

By applying 5.4 and 5.5, one obtains a continuous path of unitaries  $\{u(t) : t \in [0, 1]\}$  such that

$$u(0) = u, \quad u(1) = 1 \text{ and} \quad (\text{e 7.133})$$

$$\|u(t)^*\phi(f)u(t) - \phi(f)\| < \epsilon/2 \text{ for all } f \in \mathcal{F}. \quad (\text{e 7.134})$$

Define  $\Phi : C(X) \rightarrow C([0, 1], M_n)$  by

$$\pi_t \circ \Phi = \text{ad } u(1-t) \circ \phi \text{ for all } t \in [0, 1].$$

Then,

$$\pi_0 \circ \Phi = \phi \text{ and } \pi_1 \circ \Phi = \psi.$$

Moreover, by (e 7.134) and (e 7.130),

$$\|\psi(f) - \pi_t \circ \Phi(f)\| < \epsilon \text{ for all } f \in \mathcal{F} \text{ and } t \in [0, 1].$$

□

**Lemma 7.2.** *Let  $X$  be a finite CW complex with torsion  $K_1(C(X))$  and let  $k$  be the largest order of torsion elements in  $K_i(C(X))$  ( $i = 0, 1$ ). Let  $\epsilon > 0$ ,  $\mathcal{F} \subset C(X)$  be a finite subset and let  $\sigma > 0$ . There exist  $\eta > 0$  (which depends on  $\epsilon$  and  $\mathcal{F}$  but not  $\sigma$ ), a finite subset  $\mathcal{G} \subset C(X)$  and  $\delta > 0$  satisfying the following:*

*Suppose that  $\phi, \psi : C(X) \rightarrow M_n$  (for any integer  $n$ ) are two unital homomorphisms such that*

$$\|\phi(f) - \psi(f)\| < \delta \text{ for all } f \in \mathcal{G}, \quad (\text{e 7.135})$$

$$\mu_{\tau \circ \phi}(O_\eta) \geq \sigma\eta \text{ and } \mu_{\tau \circ \psi}(O_\eta) \geq \sigma\eta \quad (\text{e 7.136})$$

*for any open ball  $O_\eta$  of radius  $\eta$ , where  $\tau$  is the normalized trace on  $M_n$ , and*

$$\text{ad } u \circ \phi = \psi \quad (\text{e 7.137})$$

*for some unitary  $u \in A$ . Then, there exists a homomorphism  $\Phi : C(X) \rightarrow M_{k_0}(C([0, 1], M_n))$  such that*

$$\begin{aligned} \pi_0 \circ \Phi &= \phi^{(k_0)}, \quad \pi_1 \circ \Phi = \psi^{(k_0)} \quad \text{and} \\ \|\psi^{(k_0)}(f) - \pi_t \circ \Phi(f)\| &< \epsilon \text{ for all } f \in \mathcal{F}, \end{aligned}$$

*where  $k_0 = k!$ , and  $\phi^{(k_0)}(f) = \text{diag}(\overbrace{\phi(f), \phi(f), \dots, \phi(f)}^{k_0})$  and  $\psi^{(k_0)}(f) = \text{diag}(\overbrace{\psi(f), \psi(f), \dots, \psi(f)}^{k_0})$  for all  $f \in C(X)$ , respectively.*

*Proof.* By [6], one has

$$\text{Hom}_\Lambda(\underline{K}(C(X)), \underline{K}(M_n)) = \text{Hom}_\Lambda(F_k \underline{K}(C(X)), F_k \underline{K}(M_n)).$$

Let  $k_0 = k!$ . It follows that

$$\overbrace{\lambda + \lambda + \dots + \lambda}^{k_0} = 0,$$

for any homomorphism  $\lambda$  from  $K_1(C(X), \mathbb{Z}/m\mathbb{Z})$  with  $m \leq k_0$ . Thus the lemma follows from the proof of 7.1 (to  $\phi^{(k_0)}$  and  $\psi^{(k_0)}$ ). The point is that

$$\text{Bott}(\phi^{(k_0)}, u^{(k_0)})|_{\mathcal{P}'} = \{0\}$$

for any finite subset  $\mathcal{P}' \subset K_1(C(X), \mathbb{Z}/m\mathbb{Z})$  for  $0 \leq m \leq k_0$  as long as it is defined, where

$$u^{(k_0)} = \text{diag}(\overbrace{u, u, \dots, u}^{k_0}). \quad \square$$

**Lemma 7.3.** *Let  $X = \mathbb{T}$  or  $X = I \times \mathbb{T}$  (with the product metric). Let  $\mathcal{F} \subset C(X)$  be a finite subset and let  $\epsilon > 0$ . There exists  $\eta_1 > 0$  such that, for any  $\sigma_1 > 0$ , the following holds: There exists a finite subset  $\mathcal{G} \subset C(X)$  and there exists  $\eta_2 > 0$  such that, for any  $\sigma_2 > 0$ , there exists  $\delta > 0$  satisfying the following:*

*Suppose that  $\phi, \psi : C(X) \rightarrow M_n$  (for some integer  $n$ ) are two unital homomorphisms such that*

$$\|\phi(f) - \psi(f)\| < \delta \text{ for all } f \in \mathcal{G} \quad (\text{e 7.138})$$

$$\mu_{\tau \circ \phi}(O_{\eta_1}) \geq \sigma_1\eta_1, \quad \mu_{\tau \circ \psi}(O_{\eta_1}) \geq \sigma_1\eta_1, \quad (\text{e 7.139})$$

$$\mu_{\tau \circ \phi}(O_{\eta_2}) \geq \sigma_2\eta_2 \text{ and } \mu_{\tau \circ \psi}(O_{\eta_2}) \geq \sigma_2\eta_2 \quad (\text{e 7.140})$$

*for any open ball  $O_{\eta_j}$  of radius  $\eta_j$ ,  $j = 1, 2$ , where  $\tau$  is the normalized trace on  $M_n$ , and*

$$\text{ad } u \circ \phi = \psi \quad (\text{e 7.141})$$



for some unitary  $u \in M_n$ . Then, there exists a homomorphism  $\Phi : C(X) \rightarrow C([0, 1], M_n)$  such that

$$\begin{aligned} \pi_0 \circ \Phi &= \phi, \quad \pi_1 \circ \Phi = \psi \quad \text{and} \\ \|\psi(f) - \pi_t \circ \Phi(f)\| &< \epsilon \quad \text{for all } f \in \mathcal{F}. \end{aligned}$$

*Proof.* Let  $\delta_{00} > 0$  be satisfying the following: for any pair of unitaries  $u_0, v_0$  in a unital  $C^*$ -algebra,  $\text{bott}_1(u_0, v_0)$  is well defined whenever  $\|[u_0, v_0]\| < \delta_{00}$ . We will prove the case that  $X = I \times \mathbb{T}$ . The proof for the case that  $X = \mathbb{T}$  follows from the same argument but simpler. Let  $\epsilon > 0$  and  $\mathcal{F}$  be given as in the lemma. Let  $\mathcal{F}_1 = \mathcal{F} \cup \{z\}$ , where

$$z(t, e^{2\pi i s}) = e^{2\pi i s} \quad \text{for all } t \in [0, 1] \quad \text{and } s \in [0, 1].$$

Let  $\eta_1 > 0$  (in place of  $\eta$ ) be required by 5.4 for  $\epsilon/4$  (in place of  $\epsilon$ ) and  $\mathcal{F}_1$  (in place of  $\mathcal{F}$ ). Let  $\sigma_1 > 0$ .

Let  $\mathcal{G} \subset C(X)$  be a finite subset, let  $\delta_0 > 0$  (in place of  $\delta$ ) and  $\mathcal{P} \subset \underline{K}(C(X))$  be a subset required by 5.4 for  $\epsilon/4$  (in place of  $\epsilon$ ) and  $\mathcal{F}_1$  (in place of  $\mathcal{F}$ ) and  $\sigma_1/2$  (as well as for  $X = I \times \mathbb{T}$ ).

Since  $K_0(C(X)) = \mathbb{Z}$  and  $K_1(C(X)) = \mathbb{Z}$ , without loss of generality, we may assume that  $\mathcal{P} = \{[z]\}$ . We assume that  $\delta_0 < \delta_{00}/2$ . We may also assume that  $\delta_0$  satisfies the following: if  $u_1, u_2$  and  $v$  are unitaries with

$$\|u_1 - u_2\| < \delta_0 \quad \text{and} \quad \|[u_1, v]\| < \delta_0,$$

then

$$\text{bott}_1(u_1, v) = \text{bott}_1(u_2, v) \tag{e 7.142}$$

(whenever  $\|[u_1, v]\| < \delta_{00}/2$ ). Let  $\eta'_2 > 0$  such that

$$|f(x) - f(x')| < \min\{\delta_0/2, \epsilon/16\} \quad \text{for all } f \in \mathcal{G} \cup \mathcal{F} \tag{e 7.143}$$

provided that  $\text{dist}(x, y) < \eta_2$ . Choose an integer  $K > 1$  such that  $2\pi/K < \min\{\eta_1/16, \eta'_2/16\}$  and put  $\eta_2 = \pi/4K$ . Let  $\sigma_2 > 0$ . Choose  $\delta = \min\{\delta_0/2, \sigma_2\eta_2/2\}$ .

Suppose that  $\phi$  and  $\psi$  satisfy the assumption of the lemma for the above  $\mathcal{G}$ ,  $\eta_1, \eta_2, \sigma_1, \sigma_2$  and  $\delta$ . Let  $w_j = e^{2j\pi\sqrt{-1}/K}$  and  $\zeta_j = 1 \times w_j$ ,  $j = 1, 2, \dots, K$ . Then, by the assumption,

$$\mu_{\tau \circ \psi}(B_{\eta_2}(\zeta_j)) \geq \sigma_2\eta_2 > 2\delta, \tag{e 7.144}$$

$j = 1, 2, \dots, K$ . Note that

$$B_{\eta_2}(\zeta_j) \cap B_{\eta_2}(\zeta'_j) = \emptyset, \tag{e 7.145}$$

if  $j \neq j'$ ,  $j, j' = 1, 2, \dots, K$ .

Write

$$\psi(f) = \sum_{l=1}^N f(x_l) e_l \quad \text{for all } f \in C(X), \tag{e 7.146}$$

where  $\{e_1, e_2, \dots, e_N\}$  is a set of mutually orthogonal projections and  $x_1, x_2, \dots, x_N$  are distinct points in  $X$ . Define

$$p_j = \sum_{x_l \in B_{\eta_2}(\zeta_j)} e_l, \quad j = 1, 2, \dots, K.$$

By (e 7.144),

$$\tau(p_j) \geq \sigma_2 \eta_2, \quad j = 1, 2, \dots, K. \quad (\text{e 7.147})$$

Put

$$\gamma = \frac{1}{2\pi i} \tau(\log(u^* \phi(z) u \phi(z)^*)), \quad (\text{e 7.148})$$

where  $\tau$  is the normalized trace on  $M_n$ . Then

$$|\gamma| < \delta. \quad (\text{e 7.149})$$

We first assume that  $\gamma \neq 0$ . For convenience, we may assume that  $\gamma < 0$ . By the Exel formula (see [12]),  $\gamma = m/n$  for some integer  $|m| < n$ .

For each  $j$ , there is a projection  $q_j \leq p_j$  such that

$$\tau(q_j) = |\gamma| \quad \text{and} \quad q_j e_l = e_l q_j, \quad j = 1, 2, \dots, K, \quad l = 1, 2, \dots, N. \quad (\text{e 7.150})$$

There is a unitary  $v_1 \in (\sum_{j=1}^K q_j) M_n (\sum_{j=1}^K q_j)$  such that

$$v_1^* q_j v_1 = q_{j+1}, \quad j = 1, 2, \dots, K-1 \quad \text{and} \quad v_1^* q_K v_1 = q_1. \quad (\text{e 7.151})$$

Define  $v = (1 - \sum_{j=1}^K q_j) + v_1$ . Note that, by the choice of  $\delta$ , we have

$$\|[uv, \phi(f)]\| < \delta_0 \quad \text{for all } f \in \mathcal{G}. \quad (\text{e 7.152})$$

Write  $x_l = s \times e^{2\pi\sqrt{-1}t_l}$ ,  $l = 1, 2, \dots, N$ . Define  $z' = (1 - \sum_{j=1}^K q_j)\psi(z) + \sum_{j=1}^K w_j q_j$ . Then

$$\|\psi(z) - z'\| < \delta_0 \quad \text{and} \quad v^* z' v = (1 - \sum_{j=1}^K q_j)\psi(z) + \sum_{j=1}^{K-1} w_j q_{j+1} + w_K q_1. \quad (\text{e 7.153})$$

It follows that

$$\frac{1}{2\pi i} \tau(\log(v^* z' v (z')^*)) = \tau(q_j) = -\gamma. \quad (\text{e 7.154})$$

By the choice of  $\delta_0$ , we have that

$$\frac{1}{2\pi i} \tau(\log(v^* \psi(z) v \psi(z)^*)) = \tau(q_j) = -\gamma. \quad (\text{e 7.155})$$

By the choice of  $\delta_0$  (see also e 7.142) and the Exel formula, we have

$$\frac{1}{2\pi i} \tau(\log(v^* u^* \phi(z) u v \phi(z)^*)) = \frac{1}{2\pi i} \tau(\log(u^* \phi(z) u \phi(z)^*)) + \frac{1}{2\pi i} \tau(\log(v^* \phi(z) v \phi(z)^*)) \quad (\text{e 7.156})$$

$$= \frac{1}{2\pi i} \tau(\log(u^* \phi(z) u \phi(z)^*)) + \frac{1}{2\pi i} \tau(\log(v^* \psi(z) v \psi(z)^*)) \quad (\text{e 7.157})$$

$$= \gamma - \gamma = 0. \quad (\text{e 7.158})$$

It follows from the Exel formula,  $\text{bott}_1(\phi, uv) = \{0\}$  and  $\text{Bott}(\phi, uv)|_{\mathcal{P}} = \{0\}$ . It follows from 5.4 that there exists a continuous path of unitaries  $\{u(t) : t \in [0, 1/2]\}$  such that

$$u(0) = 1, \quad u(1/2) = uv \quad \text{and} \quad \|[\phi(f), u(t)]\| < \epsilon/4 \quad (\text{e 7.159})$$

for all  $f \in \mathcal{F}$  and for all  $t \in [0, 1/2]$ .

Define  $\Phi_1 : C(X) \rightarrow C([0, 1/2], M_n)$  by

$$\pi_t \circ \Phi_1(f) = u(t)^* \phi(f) u(t) \text{ for all } f \in C(X) \text{ and } t \in [0, 1/2]. \quad (\text{e 7.160})$$

Then

$$\|\pi_t \circ \Phi_1(f) - \phi(f)\| < \epsilon/2 \text{ for all } f \in \mathcal{F} \text{ and } t \in [0, 1/2]. \quad (\text{e 7.161})$$

Let

$$q_j \psi(f) = \sum_{k=1}^{N(j)} f(\xi_{k,j}) e'_{k,j} \text{ for all } f \in C(X),$$

where  $\{e'_{k,j}\}$  is a set of mutually orthogonal projections and  $\xi_{k,j} \in B_{\eta_2}(\zeta_j)$ ,  $j = 1, 2, \dots, K$ . Note that

$$v^* u^* \phi(f) uv = \psi(f) \left(1 - \sum_{j=1}^K q_j\right) + \sum_{j=1}^K \left(\sum_{k=1}^{N(j)} f(\xi_{k,j}) v_1^* e'_{k,j} v_1\right) \quad (\text{e 7.162})$$

for all  $f \in C(X)$ . It is easy to find a homomorphism  $\Phi_2 : C(X) \rightarrow C([1/2, 1], M_n)$  such that (with  $q_{K+1} = q_1$ ,  $e_{k,K+1} = e'_{k,1}$  and  $\xi_{k,K+1} = \xi_{k,1}$ )

$$\pi_{1/2} \circ \Phi_2(f) = v^* u^* \phi(f) uv, \quad (\text{e 7.163})$$

$$\pi_{3/4} \circ \Phi_2(f) = \psi(f) \left( \left(1 - \sum_{j=1}^K q_j\right) + \sum_{j=1}^K f(\zeta_j) \left(\sum_{k=1}^{N(j)} v_1^* e'_{k,j} v_1\right) \right) \quad (\text{e 7.164})$$

$$= \psi(f) \left(1 - \sum_{j=1}^{K+1} q_j\right) + \sum_{j=1}^{K-1} f(\zeta_j) q_{j+1} + f(\zeta_K) q_1 \quad (\text{e 7.165})$$

and

$$\pi_1 \circ \Phi_2(f) = \psi(f) \left(1 - \sum_{j=1}^K q_j\right) + \sum_{j=1}^{K-1} \left(\sum_{k=1}^{N(j+1)} f(\xi_{k,j+1}) e'_{k,j+1}\right) + \sum_{k=1}^{N(1)} f(\xi_{k,1}) e'_{k,1} \quad (\text{e 7.166})$$

$$= \psi(f) \quad (\text{e 7.167})$$

for all  $f \in C(X)$ . Moreover,

$$\|\pi_t \circ \Phi_2(f) - \psi(f)\| < \epsilon/16 \text{ for all } f \in \mathcal{F}. \quad (\text{e 7.168})$$

Now define  $\Phi : C(X) \rightarrow C([0, 1], M_n)$  by

$$\pi_t \circ \Phi = \pi_t \circ \Phi_1 \text{ for all } t \in [0, 1/2] \text{ and } \pi_t \circ \Phi = \pi_t \circ \Phi_2 \text{ for all } t \in [1/2, 1]. \quad (\text{e 7.169})$$

One checks that

$$\|\pi_t \circ \Phi(f) - \phi(f)\| < \epsilon \text{ for all } f \in \mathcal{F}. \quad (\text{e 7.170})$$

Finally, if  $\gamma = 0$ , we do not need  $v$  and can apply 5.4 directly.  $\square$

**Remark 7.4.** If  $\phi(f) = \sum_{l=1}^N f(x_l) p_l$  be as in the proof, let  $C_0$  be the finite dimensional commutative  $C^*$ -subalgebra generated by mutually orthogonal projections  $\{p_1, p_2, \dots, p_l\}$ . Then  $\Phi$  has the following properties:  $\pi_t \circ \Phi(f) = u(t)^* \phi(f) u(t)$  for  $t \in [0, 1/2]$ , with  $u(0) = 1$  and  $u(1) = uv$ ,  $\pi_t \circ \Phi(f) \in C_1$ , where  $C_1 \supset C_0$  is a finite dimensional commutative  $C^*$ -subalgebra.

From this, combining 6.2 and 6.3, we obtain the following which will be used in a subsequent paper:

**Lemma 7.5.** *Let  $X = \mathbb{T}$  or  $X = I \times \mathbb{T}$  (with the product metric). Let  $\mathcal{F} \subset C(X)$  be a finite subset and let  $\epsilon > 0$ . Then there exists  $\eta_1 > 0$ , for any  $\sigma_1 > 0$ , satisfying the following: There exists a finite subset  $\mathcal{G} \subset C(X)$  and there exists  $\eta_2 > 0$  such that, for any  $\sigma_2 > 0$ , there exists  $\delta > 0$  such that the following holds:*

*Suppose that  $\phi, \psi : C(X) \rightarrow M_n$  (for some integer  $n$ ) are two unital homomorphisms given by*

$$\phi(f) = \sum_{i=1}^{N_1} f(x_i)p_i \text{ and } \psi(f) = \sum_{j=1}^{N_2} f(y_j)q_j$$

*for all  $f \in C(X)$ , where  $\{x_1, x_2, \dots, x_{N_1}\}, \{y_1, y_2, \dots, y_{N_2}\} \subset X$  and where  $\{p_1, p_2, \dots, p_{N_1}\}$  and  $\{q_1, q_2, \dots, q_{N_2}\}$  are two sets of mutually orthogonal projections, such that*

$$\|\phi(f) - \psi(f)\| < \delta \text{ for all } f \in \mathcal{G} \quad (\text{e 7.171})$$

$$\mu_{\tau \circ \phi}(O_{\eta_j}) \geq \sigma_j \eta_j, \quad \mu_{\tau \circ \psi}(O_{\eta_j}) \geq \sigma_j \eta_j \quad (\text{e 7.172})$$

*for any open ball  $O_{\eta_j}$  of radius  $\eta_j$ ,  $j = 1, 2$ , where  $\tau$  is the normalized trace on  $M_n$ . Then, there exists a homomorphism  $\Phi : C(X) \rightarrow C([0, 1], M_n)$  such that*

$$\pi_0 \circ \Phi = \phi, \quad \pi_1 \circ \Phi = \psi \text{ and}$$

$$\|\psi(f) - \pi_t \circ \Phi(f)\| < \epsilon \text{ for all } f \in \mathcal{F}.$$

*Moreover,  $\pi_t \circ \Phi(C(X)) \subset C_1$  for  $t \in [0, 1/4]$ ,  $\pi_t \circ \Phi(C(X)) \subset C_2$  for  $t \in [3/4, 1]$  and*

$$\pi_t \circ \Phi(f) = u(t)^* \phi(f) u(t) \text{ for all } t \in [1/4, 3/4] \quad (\text{e 7.173})$$

*and for all  $f \in C(X)$ , where  $C_1$  is a finite dimensional commutative  $C^*$ -subalgebra containing projections  $p_1, p_2, \dots, p_{N_1}$ ,  $C_2$  is a finite dimensional commutative  $C^*$ -subalgebra containing  $q_1, q_2, \dots, q_{N_2}$ ,  $u(1/4) = 1$  and  $u(t) \in C([1/4, 3/4], M_n)$ .*

**Definition 7.6.** Let  $X$  be a compact metric space. It is said to satisfy the property (H) if the following holds.

For any finite subset  $\mathcal{F} \subset C(X)$  and for any  $\epsilon > 0$ , There exists  $\eta_1 > 0$  such that, for any  $\sigma_1 > 0$ , the following holds: There exists a finite subset  $\mathcal{G} \subset C(X)$  and  $\eta_2 > 0$ , for any  $\sigma_2 > 0$ , there exists  $\delta > 0$  satisfying the following:

Suppose that  $\phi, \psi : C(X) \rightarrow M_n$  (for any integer  $n$ ) are two unital homomorphisms such that

$$\|\phi(f) - \psi(f)\| < \delta \text{ for all } f \in \mathcal{G} \quad (\text{e 7.174})$$

$$\mu_{\tau \circ \phi}(O_{\eta_j}) \geq \sigma_j \eta_j, \quad \mu_{\tau \circ \psi}(O_{\eta_j}) \geq \sigma_j \eta_j \quad (\text{e 7.175})$$

for any open ball  $O_{\eta_j}$  of  $X$  with radius  $\eta_j$ ,  $j = 1, 2$ , where  $\tau$  is the normalized trace on  $M_n$ , and

$$\text{ad } u \circ \phi = \psi \quad (\text{e 7.176})$$

for some unitary  $u \in A$ . Then, there exists a homomorphism  $\Phi : C(X) \rightarrow C([0, 1], M_n)$  such that

$$\pi_0 \circ \Phi = \phi, \quad \pi_1 \circ \Phi = \psi \text{ and}$$

$$\|\psi(f) - \pi_t \circ \Phi(f)\| < \epsilon \text{ for all } f \in \mathcal{F}.$$

We have proved in 7.1 that if  $X$  is a finite CW complex with torsion  $K_1(C(X))$  and torsion free  $K_0(C(X))$ , then  $X$  satisfies the property (H), and have proved in 7.3 that if  $X = \mathbb{T}$  or  $X = I \times \mathbb{T}$ , then  $X$  has the property (H).

**Lemma 7.7.** Let  $X = \overbrace{\mathbb{T} \vee \mathbb{T} \vee \mathbb{T} \vee \cdots \vee \mathbb{T}}^m \vee Y$ , where  $Y$  is a finite CW complex with torsion  $K_1(C(Y))$  and torsion free  $K_0(C(Y))$ . Then  $X$  has the property (H).

*Proof.* Denote by  $\mathbb{T}_i$  the  $i$ -th copy of  $\mathbb{T}$  and denote by  $\xi_0 \in Y$  the common point of  $\mathbb{T}_i$ 's and  $Y$ . We identify  $\mathbb{T}_i$  with the unit circle and identify  $\xi_0$  with 1.

Let  $z_i \in C(X)$  be defined as follows:

$$z_i(e^{2\pi\sqrt{-1}t}) = e^{2\pi\sqrt{-1}t} \text{ for } e^{2\pi\sqrt{-1}t} \in \mathbb{T}_i \text{ and} \quad (\text{e 7.177})$$

$$z_i(x) = 1 \quad (\text{e 7.178})$$

for all other points  $x \in X$ ,  $i = 1, 2, \dots, m$ . Let  $C_0 \subset C(X)$  be a unital  $C^*$ -subalgebra which consists of those continuous functions so that it is constant on  $X \setminus (Y \setminus \{\xi_0\})$ . Note that  $C_0 \cong C(Y)$ .

Let  $\delta_{00} > 0$  be as in the proof of 7.3. Let  $\epsilon > 0$  and finite subset  $\mathcal{F} \subset C(X)$  be given. Let  $\mathcal{F}_1 = \mathcal{F} \cup \{z_1, z_2, \dots, z_m\}$ . Let  $\eta_1 > 0$  be as in the proof of 7.3 and  $\sigma_1 > 0$ . Let  $\mathcal{G} \subset C(X)$ ,  $\delta_0$ ,  $\mathcal{P} \subset \underline{K}(C(X))$  be as exactly in the proof of 7.3 (but for this  $X$ ). We may assume that  $\mathcal{P} = \{[z_1], [z_2], \dots, [z_m]\} \cup \mathcal{P}_1$  where  $\mathcal{P}_1$  can be identified with a finite subset of  $\underline{K}(C_0)$ .

Let  $\delta_0 > 0$  and  $\eta'_2 > 0$  be as in the proof of 7.3. Let  $K$  be as in the proof of 7.3. Fix an irrational number  $\theta \in (2\pi/8K, 2\pi/6K)$ . Let  $\eta_2 = \pi/16K$  and  $\sigma_2 > 0$ . Choose  $\delta = \min\{\delta_0, \sigma_2\eta_2/2\}$ .

Suppose that  $\phi$  and  $\psi$  satisfy the assumption (e 7.174), (e 7.175) and (e 7.176) for the above  $\eta_1$ ,  $\eta_2$ ,  $\sigma_1$ ,  $\sigma_2$ ,  $\mathcal{G}$ , and  $\delta$ .

Let  $w_j = e^{(2j\pi+\theta)\sqrt{-1}/K}$ ,  $j = 1, 2, \dots, K$ . Choose  $\zeta_{j,i} = w_j$  be a point in  $\mathbb{T}_i$ ,  $i = 1, 2, \dots, m$ . Then, by the assumption,

$$\mu_{\tau \circ \psi}(B_{\eta_2}(\zeta_{j,i})) \geq \sigma_2\eta_2 > 2\delta, \quad (\text{e 7.179})$$

$j = 1, 2, \dots, K$ . Note that

$$B_{\eta_2}(\zeta_{j,i}) \cap B_{\eta_2}(\zeta_{j',i'}) = \emptyset \quad (\text{e 7.180})$$

if  $j \neq j'$ ,  $j, j' = 1, 2, \dots, K$ ,  $i, i' = 1, 2, \dots, m$ . Moreover,  $1 \notin B_{\eta_2}(\zeta_{j,i})$ ,  $j = 1, 2, \dots, K$  and  $i = 1, 2, \dots, m$ .

Write

$$\psi(f) = \sum_{l=1}^N f(x_l)e_l \text{ for all } f \in C(X), \quad (\text{e 7.181})$$

where  $\{e_1, e_2, \dots, e_N\}$  is a set of mutually orthogonal projections and  $x_1, x_2, \dots, x_l$  are distinct points in  $X$ . Define

$$p_{j,i} = \sum_{x_l \in B_{\eta_2}(\zeta_{j,i})} e_l, \quad j = 1, 2, \dots, K.$$

By (e 7.179),

$$\tau(p_{j,i}) \geq \sigma_2\eta_2, \quad j = 1, 2, \dots, K \text{ and } i = 1, 2, \dots, m. \quad (\text{e 7.182})$$

Put

$$\gamma_i = \frac{1}{2\pi\sqrt{-1}} \tau(\log(u^*\phi(z_i)u\phi(z_i)^*)), \quad (\text{e 7.183})$$

where  $\tau$  is the normalized trace on  $M_n$ . Then

$$|\gamma_i| < \delta. \quad (\text{e 7.184})$$

By the Exel's formula (see [12]),  $\gamma_i = m_i/n_i$  for some integer  $|m_i| < n_i$ .

For each  $i$  and  $j$ , there is a projection  $q_{j,i} \leq p_{j,i}$  such that

$$\tau(q_{j,i}) = |\gamma_i| \text{ and } q_{j,i}e_l = e_l q_{j,i}, \quad j = 1, 2, \dots, K, \quad i = 1, 2, \dots, m \text{ and } l = 1, 2, \dots, N. \quad (\text{e 7.185})$$

There is a unitary  $v_i \in (\sum_{j=1}^K q_{j,i})M_n(\sum_{j=1}^K q_{j,i})$  such that

$$v_i^* q_{j,i} v_i = q_{j+1,i}, \quad j = 1, 2, \dots, K-1 \text{ and } v_i^* q_{K,i} v_i = q_{1,i}, \quad (\text{e 7.186})$$

if  $\gamma < 0$ , and

$$v_i^* q_{j,i} v_i = q_{j-1,i}, \quad j = 1, 2, \dots, K-1 \text{ and } v_i^* q_{1,i} v_i = q_{K,i}, \quad (\text{e 7.187})$$

if  $\gamma_i > 0$ . If  $\gamma_i = 0$ , define  $v_i = 1$ .

Define  $v = (1 - \sum_{i=1}^m \sum_{j=1}^K q_{j,i}) + \sum_{i=1}^m v_i$ . Note that, by the choice of  $\delta$ , we have

$$\|[uv, \phi(f)]\| < \delta_0 \text{ for all } f \in \mathcal{G}. \quad (\text{e 7.188})$$

Moreover, the same computation as in the proof of 7.3 shows that

$$\frac{1}{2\pi\sqrt{-1}} \tau(\log((uv)^* \phi(z_i) uv \phi(z_i)^*)) = 0, \quad i = 1, 2, \dots, m \quad (\text{e 7.189})$$

Then, using the Exel formula and 5.5, since  $K_1(C(Y))$  is torsion and  $K_0(C(Y))$  is torsion free, one obtains that

$$\text{Bott}(\phi, uv)|_{\mathcal{P}} = \{0\}. \quad (\text{e 7.190})$$

It follows from 5.4 that there exists a continuous path of unitaries  $\{u(t) : t \in [0, 1/2]\} \subset M_n$  such that

$$u(0) = uv, \quad u(1/2) = 1 \text{ and } \|\phi(f), uv\| < \epsilon/4 \quad (\text{e 7.191})$$

for all  $f \in \mathcal{F}$  and  $t \in [0, 1/2]$ . The rest of the proof is exactly the same as that of 7.3.  $\square$

**Lemma 7.8.** *Let  $X = \overbrace{\mathbb{T} \times \mathbb{T} \times \dots \times \mathbb{T}}^m$ . Then  $X$  has the property (H).*

*Proof.* Define  $z_i(e^{2\pi\sqrt{-1}t_1}, e^{2\pi\sqrt{-1}t_2}, \dots, e^{2\pi\sqrt{-1}t_m}) = e^{2\pi\sqrt{-1}t_i}$ ,  $i = 1, 2, \dots, m$ .

Let  $\delta_{00} > 0$  be as in the proof of 7.3. Let  $\epsilon > 0$ ,  $\mathcal{F} \subset C(X)$  be a finite subset be given. Let  $\mathcal{F}_1 = \mathcal{F} \cup \{z_1, z_2, \dots, z_m\}$ . Let  $\eta_1 > 0$  be as in the proof of 7.3 and let  $\sigma_1 > 0$ . Let  $\mathcal{G} \subset C(X)$ ,  $\delta_0 > 0$  and  $\mathcal{P} \subset \underline{K}(C(X))$  be as in the proof of 7.3 (for this  $X$ ).

Since  $K_0(C(X)) = \mathbb{Z}^m$  and  $K_1(C(X)) = \mathbb{Z}^m$ , by 5.5, we may assume that  $\mathcal{P} = \{[z_1], [z_2], \dots, [z_m]\}$ . Let  $\eta_2 > 0$ ,  $\sigma_2 > 0$ ,  $K$  and  $\theta$  be as in the proof 7.7.

Let  $w_j = e^{(2\pi j\sqrt{-1} + \theta)/K}$  be as in the proof of 7.7. Choose  $\zeta_{j,i} = (\overbrace{1, \dots, 1}^{i-1}, w_j, \overbrace{1, \dots, 1}^{m-i})$ ,  $j = 1, 2, \dots, K$  and  $i = 1, 2, \dots, m$ . Note that

$$B_{\eta_2}(\zeta_{j,i}) \cap B_{\eta_2}(\zeta_{j',i'}) = \emptyset \quad (\text{e 7.192})$$

if  $j \neq j'$ ,  $j, j' = 1, 2, \dots, K$ ,  $i, i' = 1, 2, \dots, m$ . Moreover,  $1 \notin B_{\eta_2}(\zeta_{j,i})$ ,  $j = 1, 2, \dots, K$  and  $i = 1, 2, \dots, m$ . Write

$$\psi(f) = \sum_{l=1}^N f(x_l) e_l \text{ for all } f \in C(X), \quad (\text{e 7.193})$$

where  $\{e_1, e_2, \dots, e_N\}$  is a set of mutually orthogonal projections and  $x_1, x_2, \dots, x_l$  are distinct points in  $X$ . Define

$$p_{j,i} = \sum_{x_l \in B_{\eta_2}(\zeta_{j,i})} e_l, \quad j = 1, 2, \dots, K.$$

By (e 7.194),

$$\tau(p_{j,i}) \geq \sigma_2 \eta_2, \quad j = 1, 2, \dots, K \quad \text{and} \quad i = 1, 2, \dots, m. \quad (\text{e 7.194})$$

Put

$$\gamma_i = \frac{1}{2\pi\sqrt{-1}} \tau(\log(u^* \phi(z_i) u \phi(z_i)^*)), \quad (\text{e 7.195})$$

where  $\tau$  is the normalized trace on  $M_n$ . Then

$$|\gamma_i| < \delta. \quad (\text{e 7.196})$$

By the Exel's formula (see [12]),  $\gamma_i = m_i/n_i$  for some integer  $|m_i| < n_i$ . For each  $i$  and  $j$ , there is a projection  $q_{j,i} \leq p_{j,i}$  such that

$$\tau(q_{j,i}) = |\gamma_i| \quad \text{and} \quad q_{j,i} e_l = e_l q_{j,i}, \quad j = 1, 2, \dots, K, \quad i = 1, 2, \dots, m \quad \text{and} \quad l = 1, 2, \dots, N. \quad (\text{e 7.197})$$

There is a unitary  $v_i \in (\sum_{j=1}^K q_{j,i}) M_n (\sum_{j=1}^K q_{j,i})$  such that

$$v_i^* q_{j,i} v_i = q_{j+1,i}, \quad j = 1, 2, \dots, K-1 \quad \text{and} \quad v_i^* q_{K,i} v_i = q_{1,i}, \quad (\text{e 7.198})$$

if  $\gamma < 0$ , and

$$v_i^* q_{j,i} v_i = q_{j-1,i}, \quad j = 1, 2, \dots, K-1 \quad \text{and} \quad v_i^* q_{1,i} v_i = q_{K,i}, \quad (\text{e 7.199})$$

if  $\gamma_i > 0$ . If  $\gamma_i = 0$ , define  $v_i = 1$ . Define  $v = (1 - \sum_{i=1}^m \sum_{j=1}^K q_{j,i}) + \sum_{i=1}^m v_i$ . Note that, by the choice of  $\delta$ , we have

$$\|[uv, \phi(f)]\| < \delta_0 \quad \text{for all } f \in \mathcal{G}. \quad (\text{e 7.200})$$

Moreover, the same computation as in the proof of 7.3 shows that

$$\frac{1}{2\pi\sqrt{-1}} \tau(\log((uv)^* \phi(z_i) uv \phi(z_i)^*)) = 0, \quad i = 1, 2, \dots, m \quad (\text{e 7.201})$$

Then, using the Exel formula and 5.5, one obtains that

$$\text{Bott}(\phi, uv)|_{\mathcal{P}} = \{0\}. \quad (\text{e 7.202})$$

It follows from 5.4 that there exists a continuous path of unitaries  $\{u(t) : t \in [0, 1/2]\} \subset M_n$  such that

$$u(0) = uv, \quad u(1/2) = 1 \quad \text{and} \quad \|\phi(f), uv\| < \epsilon/4 \quad (\text{e 7.203})$$

for all  $f \in \mathcal{F}$  and  $t \in [0, 1/2]$ . The rest of the proof is exactly the same as that of 7.3.  $\square$

**Theorem 7.9.** *Let  $X$  be a finite CW complex which has the property (H). Let  $\epsilon > 0$  be a positive number and let  $\mathcal{F}$  be a finite subset of  $C(X)$ . There exists  $\eta_1 > 0$  such that, for each  $\sigma_1 > 0$ , the following holds: There exists  $\eta_2 > 0$  such that, for any  $\sigma_2 > 0$ , there exists  $\eta_3 > 0$  such that, for any  $\sigma_3 > 0$ , there are a finite subset  $\mathcal{G} \subset C(X)$  and  $\delta > 0$  satisfying the following: Suppose that  $\phi, \psi : C(X) \rightarrow M_n$  (for some integer  $n$ ) are two unital homomorphisms such that*

$$\|\phi(f) - \psi(f)\| < \delta \text{ for all } f \in \mathcal{G}, \quad \mu_{\tau \circ \phi}(O_{\eta_j}) \geq \sigma_j \eta_j, \mu_{\tau \circ \psi}(O_{\eta_j}) \geq \sigma_j \eta_j, \quad (\text{e 7.204})$$

$$(\text{e 7.205})$$

*for any open ball  $O_{\eta_j}$  of radius  $\eta_j$ ,  $j = 1, 2, 3$  where  $\tau$  is the normalized trace. Then, there exists a homomorphism  $\Phi : C(X) \rightarrow C([0, 1], M_n)$  such that*

$$\pi_0 \circ \Phi = \phi, \quad \pi_1 \circ \Phi = \psi \text{ and}$$

$$\|\psi(f) - \pi_t \circ \Phi(f)\| < \epsilon \text{ for all } f \in \mathcal{F}.$$

*Proof.* It is clear that one can reduce the general case to the case that  $X$  is connected.

Let  $\eta'_1 > 0$  (in place of  $\eta_1$ ) be given by 7.6 for  $\epsilon/2$  and  $\mathcal{F}$ . Let  $\eta_1 = \eta'_1/16$ . Let  $\sigma_1 > 0$ . Let  $\mathcal{G}_1$  (in place of  $\mathcal{G}$ ) be a finite subset of  $C(X)$  and  $\eta'_2 > 0$  (in place of  $\eta_2$ ) be given by 7.6 for  $\eta'_1$  and  $\sigma_1/16$  (in place of  $\sigma_1$ ). Let  $\sigma_2 > 0$ .

Choose  $\delta_1$  (in place of  $\delta$ ) required by 7.6 for the given  $\epsilon/2 > 0$ ,  $\mathcal{F}$ ,  $\mathcal{G}_1$ ,  $\eta'_1$ ,  $\eta'_2$  and  $\sigma_2/16$ . We may assume that  $\eta'_2 < \eta_1$  and  $\mathcal{F} \subset \mathcal{G}$ . Denote  $\eta_2 = \eta'_2/16$ . We may assume that  $\mathcal{G}_1$  is larger than that  $\mathcal{G}$  required by 6.1 for  $\eta'_2/2$  (in place of  $\eta$ ) and  $\sigma_2/16$  (in place of  $\sigma$ ). Choose  $\delta_2 = \min\{\delta_1/2, \sigma_2\eta_2/64\}$ . Let  $\eta_0 > 0$  be such that

$$|f(x) - f(x')| < \delta_2/4 \text{ for all } f \in \mathcal{G}_1,$$

provided that  $\text{dist}(x, x') < \eta_0$ .

Let  $0 < \eta'_3 \leq \min\{\eta_0/2, \eta'_2/2\}$ . We may also assume, by choosing a smaller  $\eta_0$ , that any open ball with radius  $\eta'_3$  is path connected. Let  $\eta_3 = \eta'_3/24$  and let  $\sigma_3 > 0$ . Let  $\delta_3 > 0$  (in place of  $\delta$ ) and let  $\mathcal{G} \subset C(X)$  be a finite subset required by Lemma 6.2 for  $\delta_2/2$  (in place of  $\epsilon$ ),  $\mathcal{G}_1$  (in place of  $\mathcal{F}$ ) and  $\eta'_3$  (in place of  $\eta$ ) and  $\sigma_3/24$ . Let  $\delta = \min\{\delta_3/2, \delta_2/2\}$ .

Now suppose that  $\phi$  and  $\psi$  satisfy conditions (e 7.204) for the above  $\eta_1, \eta_2, \eta_3, \sigma_1, \sigma_2, \sigma_3, \mathcal{G}$  and  $\delta$ . In particular,

$$\mu_{\tau \circ \phi}(O_{\eta'_3/24}) \geq (\sigma_3/24)\eta'_3 \text{ and } \mu_{\tau \circ \psi}(O_{\eta'_3/24}) \geq (\sigma_3/24)\eta'_3$$

for every open ball  $O_{\eta'_3/24}$  with radius  $\eta'_3/24$ . It follows from 6.2 that there are unital homomorphisms  $\Phi_i : C(X) \rightarrow C([0, 1], M_n)$  such that

$$\pi_0 \circ \Phi_1 = \phi, \quad \pi_0 \circ \Phi_2 = \psi \quad (\text{e 7.206})$$

$$\|\pi_t \circ \Phi_1(g) - \phi(g)\| < \delta_2/2 \text{ and } \|\pi_t \circ \Phi_2(g) - \psi(g)\| < \delta_2/2 \quad (\text{e 7.207})$$

for all  $g \in \mathcal{G}_1$  and  $t \in [0, 1]$ , moreover, there is a unitary  $u \in M_n$  such that

$$\text{ad } u \circ \pi_1 \circ \Phi_1 = \pi_1 \circ \Phi_2. \quad (\text{e 7.208})$$

Note that

$$\mu_{\tau \circ \phi}(O_{\eta'_2/16}) \geq \sigma_2\eta'_2/16$$

It follows from the proof of 6.1 (with possibly larger  $\mathcal{G}$  which depends on  $\eta_2$ ) that

$$\mu_{\tau \circ \phi}(\overline{O_{\eta'_2/16}}) \leq \mu_{\tau \circ \pi_1 \circ \Phi_1}(O_{\eta_2}) \quad (\text{e 7.209})$$



It follows that

$$\mu_{\tau \circ \pi_1 \circ \Phi_1}(O_{\eta'_2}) \geq (\sigma_2/16)\eta'_2. \quad (\text{e 7.210})$$

Similarly,

$$\mu_{\tau \circ \pi_1 \circ \Phi_2}(O_{\eta_2}) \geq (\sigma_2/16)\eta_2. \quad (\text{e 7.211})$$

Moreover,

$$\mu_{\tau \circ \pi_1 \circ \Phi_1}(O_{\eta'_1}) \geq (\sigma_1/16)\eta'_1 \quad \text{and} \quad \mu_{\tau \circ \pi_1 \circ \Phi_2}(O_{\eta'_1}) \geq (\sigma_1/16)\eta'_1. \quad (\text{e 7.212})$$

Since  $X$  has the property (H), there is a unital homomorphism  $\Phi_3 : C(X) \rightarrow C([0, 1], M_n)$  such that

$$\pi_0 \circ \Phi_3 = \pi_1 \circ \Phi_1, \quad \pi_1 \circ \Phi_3 = \pi_1 \circ \Phi_2 \quad \text{and} \quad (\text{e 7.213})$$

$$\|\pi_t \circ \Phi_3(f) - \pi_1 \circ \Phi_1(f)\| < \epsilon/2 \quad \text{for all } f \in \mathcal{F}. \quad (\text{e 7.214})$$

The theorem follows from the combination of (e 7.206), (e 7.207), (e 7.213) and (e 7.214).  $\square$

## 8 Almost multiplicative maps

The purpose of this section is to prove Theorem 8.3 which states that an approximately multiplicative maps from  $C(X)$  (for those  $X$  which have the property (H)) into  $C([0, 1], M_n)$  may be approximated by homomorphisms if the  $KK$ -information they carry is the same as that of homomorphisms and they are also sufficiently injective.

The following follows from a theorem of Terry Loring ([40]).

**Theorem 8.1.** *Let  $X$  be a finite CW complex with dimension 1. Let  $\epsilon > 0$  and let  $\mathcal{F} \subset C(X)$  be a finite subset. There exists  $\delta > 0$  and a finite subset  $\mathcal{G} \subset C(X)$  satisfying the following: For any unital  $\delta$ - $\mathcal{G}$ -multiplicative contractive completely positive linear map  $\phi : C(X) \rightarrow C([0, 1], M_n)$  (for any integer  $n$ ), there is a unital homomorphism  $h : C(X) \rightarrow C([0, 1], M_n)$  such that*

$$\|\phi(f) - h(f)\| < \epsilon$$

for all  $f \in \mathcal{F}$ .

**Definition 8.2.** Let  $\mathbf{X}_0$  be the family of finite CW complexes which consists of all those with dimension no more than one and all those which have property (H). Note that  $\mathbf{X}_0$  contains all finite CW complex  $X$  with finite  $K_1(C(X))$  and torsion free  $K_0(C(X))$ ,  $I \times \mathbb{T}$ ,  $n$ -dimensional tori and those with the form  $\mathbb{T} \vee \cdots \vee \mathbb{T} \vee Y$  with some finite CW complex  $Y$  with torsion  $K_1(C(Y))$  and torsion free  $K_0(C(Y))$ .

Let  $\mathbf{X}$  be the family of finite CW complexes which contains all those in  $\mathbf{X}_0$  and those with torsion  $K_1(C(X))$ .

Let  $X$  be a finite CW complex and let  $h : C(X) \rightarrow C([0, 1], M_n)$  be a unital homomorphism. It is easy to see that there are finitely many mutually orthogonal projections  $p_1, p_2, \dots, p_m$  and points  $\xi_1, \xi_2, \dots, \xi_m$  in  $X$  with one point in each connected component such that

$$[h] = [\Phi] \quad \text{in } KK(C(X), C([0, 1], M_n)),$$

where  $\Phi(f) = \sum_{i=1}^m f(\xi_i)p_i$  for all  $f \in C(X)$ .

**Theorem 8.3.** *Let  $X \in \mathbf{X}_0$ . Let  $\epsilon > 0$  and let  $\mathcal{F} \subset C(X)$  be a finite subset. There exists  $\eta_1 > 0$  such that, for any  $\sigma_1 > 0$ , there exists  $\eta_2 > 0$  such that, for any  $\sigma_2 > 0$ , there exists  $\eta_3 > 0$  such that, for any  $\sigma_3 > 0$ , there exists a finite subset  $\mathcal{G}$ ,  $\delta > 0$ , and a finite subset  $\mathcal{P} \subset \underline{K}(C(X))$  satisfying the following:*

*Suppose that  $\phi : C(X) \rightarrow C([0, 1], M_n)$  (for any integer  $n \geq 1$ ) is a unital  $\delta$ - $\mathcal{G}$ -multiplicative contractive completely positive linear map for which*

$$\mu_{\tau \circ \phi}(O_{\eta_j}) \geq \sigma_j \eta_j \quad (\text{e 8.215})$$

*for any open ball  $O_{\eta_j}$  with radius  $\eta_j$ ,  $j = 1, 2, 3$ , and for all tracial states  $\tau$  of  $C([0, 1], M_n)$ , and*

$$[\phi]|_{\mathcal{P}} = [\Phi]|_{\mathcal{P}}, \quad (\text{e 8.216})$$

*where  $\Phi$  is a point-evaluation.*

*Then there exists a unital homomorphism  $h : C(X) \rightarrow C([0, 1], M_n)$  such that*

$$\|\phi(f) - h(f)\| < \epsilon \quad (\text{e 8.217})$$

*for all  $f \in \mathcal{F}$ .*

*Proof.* The cases that need to be considered are those  $X$  which have property (H). We may assume that  $X$  is connected and  $\Phi = \pi_\xi$  for some point  $\xi \in X$ . Let  $\epsilon > 0$  and  $\mathcal{F} \subset C(X)$  be given.

Let  $\eta_1 > 0$  be required by 7.9 for  $\epsilon/4$  (in place of  $\epsilon$ ) and  $\mathcal{F}$  above. Let  $\sigma_1 > 0$ . Let  $\eta_2 > 0$  be as required by 7.9 for  $\epsilon/4$  (in place of  $\epsilon$ ),  $\mathcal{F}$ ,  $\eta_1$  and  $\sigma_1$ . Let  $\sigma_2 > 0$ . Let  $\eta'_3 > 0$  (in place of  $\eta_3$ ) be required by 7.9 for  $\epsilon/4$  (in place of  $\epsilon$ ),  $\mathcal{F}$ ,  $\eta_1$ ,  $\eta_2$ ,  $\sigma_1$  and  $\sigma_2/4$  (in place of  $\sigma_2$ ). Let  $\sigma_3 > 0$ .

Let  $\mathcal{G}_1 \subset C(X)$  (in place of  $\mathcal{G}$ ) be a finite subset and  $\delta_1 > 0$  (in place of  $\delta$ ) required by 7.9 for  $\epsilon/4$ ,  $\mathcal{F}$ ,  $\eta_1$ ,  $\eta_2$ ,  $\eta'_3$  (in place of  $\eta_3$ ), and  $\sigma_j/4$  ( $j = 1, 2, 3$ ) as above. We may assume that  $\mathcal{F} \subset \mathcal{G}_1$ . Let  $\mathcal{G}_2 \subset C(X)$  be a finite subset which is larger than  $\mathcal{G}_1$  and which also depends on  $\eta_1$  and  $\sigma_1$ .

Let  $\epsilon_1 = \min\{\epsilon/4, \delta_1/4\}$ . Let  $\eta_3 > 0$  (in place of  $\eta$ ),  $\delta_2 > 0$  (in place of  $\delta$ ),  $\mathcal{G} \subset C(X)$  be a finite subset and  $\mathcal{P} \subset \underline{K}(C(X))$  be a finite subset required by 4.3 for  $\epsilon_1$  (in place of  $\epsilon$ ),  $\mathcal{G}_2$  (in place of  $\mathcal{F}$ ),  $\sigma_2$  (in place of  $\sigma_1$ ),  $\sigma_3/2$  (in place of  $\sigma$ ) and  $\eta_2$  (in place of  $\eta_1$ ). We may assume that  $\eta_3 < \min\{\eta'_3/2, \eta_2/2\}$ .

Suppose that  $\phi$  satisfies the assumption of the theorem for the above  $\eta_j$ ,  $\sigma_j$  ( $j = 1, 2, 3$ ),  $\delta$ ,  $\mathcal{G}$  and  $\mathcal{P}$ . Consider  $\pi_t \circ \phi$  for each  $t \in [0, 1]$ . Note that  $\underline{K}(C([0, 1], M_n)) = \underline{K}(M_n)$ . It follows that

$$[\pi_t \circ \phi]|_{\mathcal{P}} = [\pi_\xi]|_{\mathcal{P}}. \quad (\text{e 8.218})$$

Note that

$$\mu_{\tau \circ \phi}(O_{\eta_3}) \geq \sigma_3 \eta_3 \text{ and } \mu_{\tau \circ \phi}(O_{\eta_2}) \geq \sigma_2 \eta_2 \quad (\text{e 8.219})$$

for all open balls  $O_{\eta_3}$  with radius  $\eta_3$ , all open balls  $O_{\eta_2}$  with radius  $\eta_2$  and for all tracial states  $\tau$  of  $C([0, 1], M_n)$ .

By applying 4.3, one obtains, for each  $t \in [0, 1]$ , a unital homomorphism  $h_t : C(X) \rightarrow M_n$  such that

$$\|\pi_t \circ \phi(g) - h_t(g)\| < \delta_1/4 \text{ for all } g \in \mathcal{G}_1 \quad (\text{e 8.220})$$

$$\mu_{\tau \circ h_t}(O_{\eta_3}) \geq (\sigma_3/2)\eta_3 \text{ and } \mu_{\tau \circ h_t}(O_{\eta_2}) \geq (\sigma_2/2)\eta_2, \quad (\text{e 8.221})$$

where  $\tau$  is the unique tracial state on  $M_n$ . Note that, by choosing the large  $\mathcal{G}_2$  (depends on  $\epsilon_1$  and  $\sigma_1$ ) and smaller  $\delta_1$ , we may also assume that

$$\mu_{\tau \circ h_t}(O_{\eta_1}) \geq (\sigma_1/2)\eta_1. \quad (\text{e 8.222})$$

There is a partition  $0 = t_0 < t_1 < \dots < t_m = 1$  such that

$$\|\pi_{t_i} \circ \phi(g) - \pi_{t_{i-1}} \circ \phi(g)\| < \delta_1/4 \text{ for all } g \in \mathcal{G}_1, \quad (\text{e 8.223})$$

$i = 1, 2, \dots, m$ . Therefore

$$\|h_{t_i}(g) - h_{t_{i-1}}(g)\| < \|h_{t_i}(g) - \pi_{t_i} \circ \phi(g)\| \quad (\text{e 8.224})$$

$$+ \|\pi_{t_i} \circ \phi(g) - \pi_{t_{i-1}} \circ \phi(g)\| + \|\pi_{t_{i-1}} \circ \phi(g) - h_{t_{i-1}}(g)\| \quad (\text{e 8.225})$$

$$< \delta_1/4 + \delta_1/4 + \delta_1/4 < \delta_1 \quad (\text{e 8.226})$$

for all  $g \in \mathcal{G}_1$ . Thus, using (e 8.219) and (e 8.222), and, by applying 7.9, there exists, for each  $i$ , a unital homomorphism  $\Phi_i : C(X) \rightarrow C([t_{i-1}, t_i], M_n)$  such that

$$\pi_{t_{i-1}} \circ \Phi_i = h_{t_{i-1}}, \quad \pi_{t_i} \circ \Phi_i = h_{t_i} \text{ and } \|\pi_t \circ \Phi_i(f) - h_{t_i}(f)\| < \epsilon/4 \quad (\text{e 8.227})$$

for all  $f \in \mathcal{F}$ ,  $i = 1, 2, \dots, m$ .

Define  $h : C(X) \rightarrow C([0, 1], M_n)$  by

$$\pi_t \circ h = \pi_t \circ \Phi_i \text{ if } t \in [t_{i-1}, t_i],$$

$i = 1, 2, \dots, m$ . It follows that

$$\|h(f) - \phi(f)\| < \epsilon \text{ for all } f \in \mathcal{F}.$$

□

**Lemma 8.4.** *Let  $X \in \mathbf{X}$ . Let  $\epsilon > 0$  and  $\mathcal{F} \subset C(X)$  be a finite subset. Suppose that  $k_0 = k!$ , where  $k$  is the largest finite order of torsion elements in  $K_i(C(X))$ ,  $i = 0, 1$ .*

*There exists  $\eta_1 > 0$  such that, for any  $\sigma_1 > 0$ , there exists  $\eta_2 > 0$  such that, for any  $\sigma_2 > 0$ , there exists  $\eta_3 > 0$  such that, for any  $\sigma_3 > 0$ , the following holds: There is a finite subset  $\mathcal{G} \subset C(X)$ , there is  $\delta > 0$  and there is a finite subset  $\mathcal{P} \subset \underline{K}(C(X))$  satisfying the following:*

*Suppose that  $\phi : C(X) \rightarrow C([0, 1], M_n)$  is a unital  $\delta$ - $\mathcal{G}$ -multiplicative contractive completely positive linear map for which*

$$\mu_{\tau \circ \phi}(O_{\eta_j}) \geq \sigma_j \eta_j \quad (\text{e 8.228})$$

*for any open ball  $O_{\eta_j}$  with radius  $\eta_j$ ,  $j = 1, 2, 3$ , and for all tracial states  $\tau$  of  $C([0, 1], M_n)$ , and*

$$[\phi]|_{\mathcal{P}} = [\Phi]|_{\mathcal{P}}, \quad (\text{e 8.229})$$

*where  $\Phi$  is a point-evaluation.*

*Then there exists a unital homomorphism  $h : C(X) \rightarrow M_{k_0}(C([0, 1], M_n))$  such that*

$$\|\phi^{(k_0)}(f) - h(f)\| < \epsilon \quad (\text{e 8.230})$$

*for all  $f \in \mathcal{F}$ , where  $\phi^{(k_0)}(f) = \text{diag}(\overbrace{\phi(f), \phi(f), \dots, \phi(f)}^{k_0})$  for all  $f \in C(X)$ .*

*Proof.* The proof is exactly the same as that of 8.3 but applying 7.2 instead. □

**Corollary 8.5.** *Let  $X \in \mathbf{X}_0$ . Let  $\epsilon > 0$ , let  $\mathcal{F} \subset C(X)$  be a finite subset and  $\Delta : (0, 1) \rightarrow (0, 1)$  be a non-decreasing map. There exists  $\eta > 0$ , a finite subset  $\mathcal{G}$ ,  $\delta > 0$ , and a finite subset  $\mathcal{P} \subset \underline{K}(C(X))$  satisfying the following:*

Suppose that  $\phi : C(X) \rightarrow C([0, 1], M_n)$  (for any integer  $n \geq 1$ ) is a unital  $\delta$ - $\mathcal{G}$ -multiplicative contractive completely positive linear map for which

$$\mu_{\tau \circ \phi}(O_a) \geq \Delta(a) \quad (\text{e 8.231})$$

for any open ball  $O_a$  with radius  $a \geq \eta$  and for all tracial states  $\tau$  of  $C([0, 1], M_n)$ , and

$$[\phi]|_{\mathcal{P}} = [\Phi]|_{\mathcal{P}}, \quad (\text{e 8.232})$$

where  $\Phi$  is a point-evaluation.

Then there exists a unital homomorphism  $h : C(X) \rightarrow C([0, 1], M_n)$  such that

$$\|\phi(f) - h(f)\| < \epsilon \quad (\text{e 8.233})$$

for all  $f \in \mathcal{F}$ .

*Proof.* Let  $\epsilon > 0$ ,  $\mathcal{F} \subset C(X)$  be a finite subset and  $\Delta$  be given as described. Let  $\eta_1 > 0$  be as required by 8.3. Let  $\sigma_1 = \Delta(\eta_1)/\eta_1$ . Let  $\eta_2 > 0$  be required by 8.3 for the above  $\epsilon$ ,  $\mathcal{F}$ ,  $\eta_1$  and  $\sigma_2$ . Let  $\sigma_2 = \Delta(\eta_2)/\eta_2$ . Let  $\eta_3 > 0$  be required by the above  $\epsilon$ ,  $\mathcal{F}$ ,  $\eta_j$  and  $\sigma_j$ ,  $j = 1, 2$ . Let  $\sigma_3 = \Delta(\eta_3)/\eta_3$ . Choose  $\eta = \min\{\eta_j : j = 1, 2, 3\}$ . We then choose  $\delta > 0$ ,  $\mathcal{G}$  and  $\mathcal{P}$  as required by 8.3 for the above  $\epsilon$ ,  $\mathcal{F}$ ,  $\eta_j$  and  $\sigma_j$ ,  $j = 1, 2, 3$ . Suppose that  $\phi$  satisfies the assumption for the above  $\eta$ ,  $\delta$ ,  $\mathcal{G}$  and  $\mathcal{P}$ . Then  $\phi$  satisfies the assumption of 8.3 for the above  $\eta_j$ ,  $\sigma_j$ ,  $\delta$  and  $\mathcal{P}$ . We then apply 8.3. □

**Remark 8.6.** Note that 8.4 also has its version of 8.5.

## 9 Simple $C^*$ -algebras of tracial rank one

This section collects a number of elementary facts about simple  $C^*$ -algebras with tracial rank one.

**9.1.** Let  $B = \oplus_{j=1}^m C(X_j, M_{r(j)})$ , where  $X_j = [0, 1]$  or  $X_j$  is a point. For  $j \leq m$ , denote by  $t_{j,x}$  the normalized trace at  $x \in X_j$  for the  $j$ -th summand, i.e., if  $b \in B$ , then

$$t_{j,x}(b) = \tau(\pi_j(b)(x)),$$

where  $\pi_j : B \rightarrow C([0, 1], M_{r(j)})$  is the projection to the  $j$ -th summand,  $x \in X_j$  and  $\tau$  is the normalized trace on  $M_{r(j)}$ .

**Lemma 9.2.** Let  $A$  be a unital simple separable  $C^*$ -algebra with tracial rank one or zero, let  $\epsilon > 0$ , let  $\eta > 0$ , let  $0 < r < 1$ , let  $\mathcal{F} \subset A$  be a finite subset and let  $a_1, a_2, \dots, a_l, b_1, b_2, \dots, b_m \in A_+ \setminus \{0\}$  be such that

$$\tau(a_i) \geq \sigma_i \text{ and } \tau(b_j) \leq d_j, \text{ for all } \tau \in T(A) \quad (\text{e 9.234})$$

for some  $\sigma_i > 0$  and  $d_j > 0$ ,  $i = 1, 2, \dots, l$  and  $j = 1, 2, \dots, m$ .

Then there exists a projection  $p \in A$  and a  $C^*$ -subalgebra  $B = \oplus_{k=1}^K C(X_k, M_{r(j)})$ , where  $X_k = [0, 1]$  or  $X_k$  is a point, with  $1_B = p$  such that

$$\|pc - cp\| < \epsilon, \text{ dist}(pcp, B) < \epsilon \text{ for all } c \in \mathcal{F}, \quad (\text{e 9.235})$$

$$\tau(1 - p) < \eta \text{ for all } \tau \in T(A), \quad (\text{e 9.236})$$

$$t_{k,x}(L(a_i)) \geq r \cdot \sigma_i \text{ and } t_{k,x}(L(b_j)) \leq \frac{1}{r} d_j, \quad (\text{e 9.237})$$

for  $i = 1, 2, \dots, l$ ,  $j = 1, 2, \dots, m$ , for each  $x \in X_k$  and  $k = 1, 2, \dots, K$ , for each normalized trace  $t_{j,x}$  at each  $x \in X_j$ , for each of the  $j$ -th summand of  $B$ , and for any  $L(a_i), L(b_j) \in B_+$  with

$$\|L(a_i) - pa_i p\| < \epsilon \text{ and } \|L(b_j) - pb_j p\| < \epsilon \quad (\text{e 9.238})$$

*Proof.* There exists of a sequence of projections  $p_n \in A$  such that

$$\lim_{n \rightarrow \infty} \|cp_n - p_n c\| = 0 \text{ for all } c \in A, \quad (\text{e 9.239})$$

and there exists a sequence of  $C^*$ -subalgebras  $B_n = \bigoplus_{j=1}^{m(n)} C(X_{j,n}, M_{r(j,n)})$  (where  $X_{j,n} = [0, 1]$  or  $X$  is a single point) such that

$$\lim_{n \rightarrow \infty} \text{dist}(p_n c p_n, B_n) = 0 \text{ and } \lim_{n \rightarrow \infty} \sup_{\tau \in T(A)} \{\tau(1 - p_n)\} = 0. \quad (\text{e 9.240})$$

There exists a contractive completely positive linear map  $L_n : p_n A p_n \rightarrow B_n$  such that

$$\lim_{n \rightarrow \infty} \|L_n(a) - p_n a p_n\| = 0 \text{ for all } a \in A$$

(see 2.3.9 of [22]).

Let  $1 > r > 0$ . Suppose that there exists  $i$  (or  $j$ ) and there exists a subsequence  $\{n_k\}$ ,  $\{j_k\}$  and  $\{x_k\} \in [0, 1]$  such that

$$t_{j_k, x_k}(\pi_{j_k}(L_k(a_i))) < r \cdot \sigma_i \text{ (or } t_{j_k, x_k}(\pi_{j_k}(L_k(b_j))) > \frac{1}{r} d_j) \quad (\text{e 9.241})$$

for all  $k$ . Define a state  $T_k : A \rightarrow \mathbb{C}$  by  $T_k(a) = t_{j_k, x_k}(a)$ ,  $k = 1, 2, \dots$ . Let  $T$  be a limit point. Note  $T_k(1_A) = 1$ . Therefore  $T$  is a state on  $A$ . Then, by (e 9.241),

$$T(a_i) \leq r \cdot \sigma_i \text{ (or } T(b_j) \geq \frac{1}{r} \cdot d_j) \quad (\text{e 9.242})$$

However, it is easy to check that  $T$  is a tracial state. This contradicts with (e 9.234). The lemma follows by choosing  $p$  to be  $p_n$  and  $B$  to be  $B_n$  for some sufficiently large  $n$ .  $\square$

**Lemma 9.3.** *Let  $\sigma > 0$  and let  $1 > r > 0$ . There exists  $\delta > 0$  for any pair of  $a, b \in A_+ \setminus \{0\}$  with  $0 \leq a, b \leq 1$ , where  $A$  is a unital separable simple  $C^*$ -algebra with tracial rank one or zero,*

$$\|ab - b\| < \delta \text{ and } \tau(b) \geq \sigma \text{ for all } \tau \in T(A), \quad (\text{e 9.243})$$

*there exists a projection  $e \in \overline{aAa}$  such that*

$$\tau(e) > r\sigma \text{ for all } \tau \in T(A). \quad (\text{e 9.244})$$

*Proof.* For any  $1 > d > 0$ , define a function  $f_d \in C([0, \infty))$  as follows:

$$f_d(t) = \begin{cases} 0, & \text{if } 0 < t < d/2; \\ \text{linear}, & \text{if } d/2 \leq t < d; \\ 1, & \text{if } d \leq t < \infty \end{cases}$$

Choose  $1 > r_1 > r > 0$ . There exists  $\epsilon > 0$  such that, for any  $0 \leq c \leq 1$ ,

$$\|f_\epsilon(c)c - c\| < (r_1 - r)\sigma/8, \quad (\text{e 9.245})$$

Note that  $\epsilon$  is independent of  $c$ . There exists  $\delta > 0$  such that if  $0 \leq a_1, b_1 \leq 1$ ,

$$\|a_1 b_1 - b_1\| < \delta, \quad (\text{e 9.246})$$

then

$$\|a_1 f_{\epsilon/2}(b_1) - f_{\epsilon/2}(b_1)\| < 1/16. \quad (\text{e 9.247})$$

Moreover, there exists  $\delta_1 > 0$ , if  $0 \leq a_1, a_2 \leq 1$  and

$$\|a_1 - a_2\| < \delta_1, \quad (\text{e 9.248})$$

then

$$\|f_{\epsilon/2}(a_1) - f_{\epsilon/2}(a_2)\| < \delta/2. \quad (\text{e 9.249})$$

For any  $0 < \eta < \min\{\delta_1/2, \delta/2, (\frac{r_1-r}{8})\sigma\}$ , by applying 9.2, one chooses a projection  $p \in A$  such that there exists a  $C^*$ -subalgebra  $B = \oplus_{j=1}^m C(X_j, M_{r(j)})$ , where  $X_j = [0, 1]$  or  $X$  is a point, with  $1_B = p$ ,

$$\|pb - bp\| < \eta \quad (\text{e 9.250})$$

$$\text{dist}(pbp, B) < \epsilon, \quad (\text{e 9.251})$$

$$\tau(1 - p) < \eta \text{ for all } \tau \in T(A) \text{ and} \quad (\text{e 9.252})$$

$$t_{j,x}(L(b)) \geq r_1 \cdot \sigma. \quad (\text{e 9.253})$$

for each normalized trace  $t_{j,x}$  at each  $x \in [0, 1]$  for each of the  $j$ -th summand of  $B$ , and for any  $L(b) \in B_+$  with

$$\|L(b) - eae\| < \eta. \quad (\text{e 9.254})$$

Fix such  $L(a)$ . Then, by (e 9.245) and (e 9.253),

$$t_{j,x}(f_{\epsilon}(L(b))) \geq (\frac{7r_1 + r}{8})\sigma. \quad (\text{e 9.255})$$

for all  $j$  and  $x$ . Note that, since  $A$  is simple, by Proposition 3.4 of [26], we may assume that

$$r(j) \geq 8/(r_1 - r)\sigma \quad j = 1, 2, \dots, m.$$

It follows from Lemma C of [1] that there is a projection  $e' \in B$  such that

$$e' f_{\epsilon/2}(L(b)) = e' \text{ and} \quad (\text{e 9.256})$$

$$t_{j,x}(e') \geq (\frac{7r_1 + r}{8})\sigma - (\frac{r_1 - r}{8})\sigma \quad (\text{e 9.257})$$

$$= (\frac{6r_1 + r}{8})\sigma \quad (\text{e 9.258})$$

for all  $j$  and  $x$ .

By the choice of  $\eta$ , one computes that

$$\tau(e') \geq r\sigma \text{ for all } \tau \in T(A). \quad (\text{e 9.259})$$

Put  $c = (1 - p)b(1 - p) + L(b)$ . Then

$$\|b - c\| < \delta_1. \quad (\text{e 9.260})$$

It follows from (e 9.249) and (e 9.247) that

$$\|af_\epsilon(c) - f_\epsilon(c)\| < 1/16. \quad (\text{e 9.261})$$

Thus

$$\|ae' - e'\| < 1/16. \quad (\text{e 9.262})$$

Therefore

$$\|ae'a - e'\| < 1/8. \quad (\text{e 9.263})$$

It follows (for example, Lemma 2.5.4 of [22]) that there exists a projection  $e$  in the  $C^*$ -subalgebra generated by  $ae'a$  such that

$$\|e - e'\| < 1/4. \quad (\text{e 9.264})$$

Therefore

$$\tau(e) = \tau(e') \geq r\sigma \text{ for all } \tau \in T(A). \quad (\text{e 9.265})$$

Moreover,  $e$  is in  $\overline{aAa}$ .

□

**Corollary 9.4.** *Let  $A$  be a unital simple separable  $C^*$ -algebra with tracial rank one or zero and let  $a \in A_+ \setminus \{0\}$  with  $\|a\| \leq 1$ . Suppose that*

$$\tau(a) \geq \sigma \text{ for all } \tau \in T(A) \quad (\text{e 9.266})$$

*for some  $\sigma > 0$ . Then, for any  $1 > r > 0$ , there is a projection  $e \in \overline{aAa}$  such that*

$$\tau(e) \geq r\sigma \text{ for all } \tau \in T(A). \quad (\text{e 9.267})$$

*Proof.* For any  $b \in A_+$  and any  $\delta > 0$ , there exists  $\epsilon > 0$  such that

$$\|f_\epsilon(b)b - b\| < \delta,$$

where  $f_\epsilon$  is as defined in the proof of 9.3. Then one sees that the corollary follows immediately from the previous lemma.

□

**Proposition 9.5.** *Let  $A$  be a unital separable simple  $C^*$ -algebra with tracial rank no more than one and let  $p \in A$  be a projection. Then, for any  $\sigma > 0$  and integers  $m > n \geq 1$ , there exists a projection  $q \leq p$  such that*

$$\frac{n+1}{m}\tau(p) > \tau(q) > \frac{n}{m}\tau(p) \text{ for all } \tau \in T(A). \quad (\text{e 9.268})$$

*Proof.* This follows from the fact that  $A$  is tracially approximately divisible (see 2.16) .

□

**Lemma 9.6.** *Let  $X$  be a compact metric space, let  $\Delta : (0, 1) \rightarrow (0, 1)$  be a non-decreasing map, let  $\epsilon > 0$ , let  $\mathcal{F} \subset C(X)$  be a finite subset and let  $\{x_1, x_2, \dots, x_m\}$  be a finite subset. Let  $\eta > 0$  be such that*

$$|f(x) - f(x')| < \epsilon/4 \text{ for all } f \in \mathcal{F},$$

*if  $\text{dist}(x, x') < 2\eta$  and*

$$O_{2\eta}(x_i) \cap O_{2\eta}(x_j) = \emptyset \text{ if } i \neq j$$

*(so  $\eta$  does not depend on  $\Delta$ ). Let  $1 > r > 0$ . Then there exists  $\delta > 0$  and a finite subset  $\mathcal{G} \subset C(X)$  satisfying the following:*

*For any unital separable simple  $C^*$ -algebra  $A$  with tracial rank no more than one and any unital  $\delta$ - $\mathcal{G}$ -multiplicative contractive completely positive linear map  $L : C(X) \rightarrow A$  for which*

$$\mu_{\tau \circ L}(O_a) \geq \Delta(a) \text{ for all } \tau \in T(A) \quad (\text{e 9.269})$$

*and for all  $1 > a \geq \eta$ , there exist mutually orthogonal non-zero projections  $p_1, p_2, \dots, p_m$  in  $A$  such that*

$$\tau(p_i) \geq r\Delta(\eta) \text{ for all } \tau \in T(A), \ i = 1, 2, \dots, m \text{ and} \quad (\text{e 9.270})$$

$$\|L(f) - [PL(f)P + \sum_{i=1}^m f(x_i)p_i]\| < \epsilon \text{ for all } f \in \mathcal{F}, \quad (\text{e 9.271})$$

*where  $P = 1 - \sum_{i=1}^m p_i$ .*

*Proof.* Suppose that the lemma is false (for the above  $\epsilon, \mathcal{F}, \Delta$  and  $\{x_1, x_2, \dots, x_m\}$ ).

Let  $\eta > 0$  be such that

$$|f(x) - f(x')| < \epsilon/4 \text{ for all } f \in \mathcal{F}, \quad (\text{e 9.272})$$

if  $\text{dist}(x, x') < 2\eta$ . We may assume that  $O_{2\eta}(x_i) \cap O_{2\eta}(x_j) = \emptyset, i \neq j, i, j = 1, 2, \dots, m$ .

Let  $g_i$  be a function in  $C(X)$  such that  $0 \leq g_i(x) \leq 1$  for all  $x \in X$ ,  $g_i(x) = 1$  if  $\text{dist}(x, x_i) < \eta$  and  $g_i(x) = 0$  if  $\text{dist}(x, x_i) \geq 2\eta, i = 1, 2, \dots, m$ . Put  $\mathcal{G}_0 = \{g_i : i = 1, 2, \dots, m\}$ .

Then, there exists a sequence of unital separable simple  $C^*$ -algebras with tracial rank no more than one and a sequence of  $\delta_n$ - $\mathcal{G}_n$ -multiplicative contractive completely positive linear map  $L_n : C(X) \rightarrow A_n$  for a sequence of decreasing positive numbers  $\delta_n \rightarrow 0$  and a sequence of finite subsets  $\{\mathcal{G}_n\}$  with  $\cup_{n=1}^\infty \mathcal{G}_n$  is dense in  $C(X)$  such that

$$\mu_{\tau \circ L_n}(O_a) \geq \Delta(a) \text{ for all } \tau \in T(A) \text{ and for all } 1 > a \geq \eta \quad (\text{e 9.273})$$

$$\liminf_n \{ \inf \{ \max \{ \|L_n(f) - [P_n L_n(f) P_n + \sum_{i=1}^m f(x_i) p_{i,n}] \| : f \in \mathcal{F} \} \} \} \geq \epsilon, \quad (\text{e 9.274})$$

where infimum is taken among all possible mutually orthogonal non-zero projections  $p_{1,n}, p_{2,n}, \dots, p_{m,n}$  with  $\tau(p_{i,n}) \geq r\Delta(\eta)$  for all  $\tau \in T(A_n)$  and  $P_n = 1_{A_n} - \sum_{i=1}^n p_{i,n}$  in  $A_n$ .

Let  $B = \prod_{n=1}^\infty A_n$ , let  $Q = B / \oplus_{n=1}^\infty A_n$  and  $\Pi : B \rightarrow Q$  be the quotient map. Define  $\Phi : C(X) \rightarrow B$  by  $\Phi(f) = \{L_n(f)\}$  and  $\phi = \Pi \circ \Phi$ . Then  $\phi : C(X) \rightarrow Q$  is a unital homomorphism.

By (e 9.273),

$$\tau(L_n(g_i)) \geq \mu_{\tau \circ L_n}(O_\eta) \geq \Delta(\eta).$$

for all  $\tau \in T(A)$ . It follows from 9.4, there exists a projection  $p'_{i,n} \in \overline{L_n(g_i) A L_n(g_i)}$  such that

$$\tau(p'_{i,n}) \geq r\Delta(\eta) \text{ for all } \tau \in T(A_n), \ i = 1, 2, \dots, m. \quad (\text{e 9.275})$$



for all  $n \geq n_0$  for some  $n_0 \geq 1$ . Define  $P_i = \{p'_{i,n}\}$  (with  $p'_{i,n} = 1$  for  $n = 1, 2, \dots, n_0$ ), and  $q_i = \Pi(P_i)$ ,  $i = 1, 2, \dots, m$ . Note that

$$q_i \in \overline{\phi(g_i)A\phi(g_i)}, \quad i = 1, 2, \dots, m. \quad (\text{e 9.276})$$

It follows from Lemma 3.2 of [19] that

$$\|\phi(f) - [q\phi(f)q + \sum_{i=1}^m f(x_i)q_i]\| < \epsilon/2 \text{ for all } f \in \mathcal{F}, \quad (\text{e 9.277})$$

where  $q = 1 - \sum_{i=1}^m q_i$ . It follows that, for some sufficiently large  $n_1 \geq n_0$ ,

$$\|L_n(f) - [P_n L_n(f) P_n + \sum_{i=1}^m f(x_i) p'_{i,n}]\| < \epsilon \text{ for all } f \in \mathcal{F} \quad (\text{e 9.278})$$

for all  $n \geq n_1$ , where  $P_n = \sum_{i=1}^m p'_{i,n}$ . By (e 9.275), this contradicts with (e 9.274).  $\square$

**Lemma 9.7.** *Let  $X$  be a connected finite CW complex, let  $\xi \in X$  be a point and let  $Y = X \setminus \{\xi\}$ . Suppose that  $K_0(C_0(Y)) = \mathbb{Z}^k \oplus \text{Tor}(K_0(C_0(Y)))$  and  $g_1, g_2, \dots, g_k$  are generators of  $\mathbb{Z}^k$ . Suppose that  $\phi : C(X) \rightarrow A$  (for some unital separable simple  $C^*$ -algebra with tracial rank one or zero) is a  $\delta$ - $\mathcal{G}$ -multiplicative contractive completely positive linear map for which  $[\phi](g_i)$  is well defined ( $i = 1, 2, \dots, k$ ), where  $\delta$  is a positive number and  $\mathcal{G}$  is a finite subset of  $C(X)$ , and*

$$|\tau([\phi](g_i))| < \sigma \text{ for all } \tau \in T(A), \quad i = 1, 2, \dots, k \quad (\text{e 9.279})$$

for some  $1 > \sigma > 0$ . Then, for any  $\epsilon > 0$  and any finite subset  $\mathcal{F}$ , any  $1 > r > 0$  and any finite subset  $\mathcal{H} \subset A$ , there exists a projection  $p \in A$  and a unital  $C^*$ -subalgebra  $B = \bigoplus_{j=1}^m C(X_j, M_{r(j)})$ , where  $X_j = [0, 1]$ , or  $X_j$  is a single point, with  $1_B = p$  and a unital  $(\delta + \epsilon)$ - $\mathcal{G}$ -multiplicative contractive completely positive linear map  $L : C(X) \rightarrow B$  such that

$$\|\phi(f) - [(1-p)\phi(f)(1-p) + L(f)]\| < \epsilon \text{ for all } f \in \mathcal{F} \text{ and} \quad (\text{e 9.280})$$

$$|t_{j,x}([L](g_i))| < (1+r)\sigma \quad j = 1, 2, \dots, k \text{ and } x \in X_j. \quad (\text{e 9.281})$$

(We use  $t_{j,x}$  for  $\tau_{j,x} \otimes \text{Tr}_R$  on  $B \otimes M_R$ , where  $\text{Tr}_R$  is the standard trace on  $M_R$ .) Moreover,

$$\|pa - ap\| < \epsilon \text{ for all } a \in \mathcal{H}.$$

*Proof.* The proof is similar to that of 9.2. Let  $p_j, q_j \in M_R(C(X))$  such that

$$[p_j] - [q_j] = g_j, \quad j = 1, 2, \dots, k$$

for some integer  $R \geq 1$ . There exists of a sequence of projections  $p_n \in A$  such that

$$\lim_{n \rightarrow \infty} \|cp_n - p_nc\| = 0 \text{ for all } c \in A, \quad (\text{e 9.282})$$

and there exists a sequence of  $C^*$ -subalgebras  $B_n = \bigoplus_{j=1}^{m(n)} C(X_{j,n}, M_{r(j,n)})$  (where  $X_{j,n} = [0, 1]$  or  $X$  is a single point) with  $1_{B_n} = p_n$  such that

$$\lim_{n \rightarrow \infty} \text{dist}(p_n c p_n, B_n) = 0 \text{ and } \lim_{n \rightarrow \infty} \sup_{\tau \in T(A)} \{\tau(1 - p_n)\} = 0. \quad (\text{e 9.283})$$

For sufficiently large  $n$ , there exists a contractive completely positive linear map  $L'_n : p_n A p_n \rightarrow B_n$  such that

$$\lim_{n \rightarrow \infty} \|L'_n(a) - p_n a p_n\| = 0 \text{ for all } a \in A.$$

(see 2.3.9 of [22]). We have

$$\lim_{n \rightarrow \infty} \|\phi(f) - [(1 - p_n)\phi(f)(1 - p_n) + L'_n \circ \phi(f)]\| = 0 \text{ for all } f \in C(X). \quad (\text{e 9.284})$$

Define  $L'_{n,R} : M_R(A) \rightarrow M_R(A)$  by  $L'_{n,R} \otimes \text{id}_{M_R}$  and  $\phi_R : M_R(C(X)) \rightarrow M_R(A)$  by  $\phi_R = \phi \otimes \text{id}_{M_R}$ .

Suppose that (for some fixed  $1 > r > 0$ ) there exists a subsequence  $\{n_k\}$ ,  $\{j_k\}$  and  $\{x_k\} \in [0, 1]$  such that

$$t_{j_k, x_k}(L'_{n_k, R} \circ \phi_R(p_i - q_i)) \geq (1 + r)\sigma \quad (\text{e 9.285})$$

for all  $k$ . Define a state  $T_k : A \rightarrow \mathbb{C}$  by  $T_k(a) = t_{j_k, x_k}(a)$ ,  $k = 1, 2, \dots$ . Let  $T$  be a limit point. Note  $T_k(1_A) = 1$ . Therefore  $T$  is a state on  $A$ . Then, by (e 9.285),

$$T([\phi](g_i)) \geq (1 + r)\sigma. \quad (\text{e 9.286})$$

However, it is easy to check that  $T$  is a tracial state. This contradicts with (e 9.279). So the lemma follows by choosing  $B$  to be  $B_n$ ,  $p$  to be  $p_n$  and  $L$  to be  $L'_n \circ L$  for some sufficiently large  $n$ . □

**Lemma 9.8.** *Let  $A$  be a unital separable simple  $C^*$ -algebra with tracial rank no more than one. Let  $p_1, p_2, \dots, p_n$  be a finite subset of projections in  $A$ , and let  $L : C(X) \rightarrow A$  be a contractive completely positive linear map with  $L(1_{C(X)})$  being a projection. Let  $d_1, d_2, \dots, d_n$  be positive numbers and  $\Delta : (0, 1) \rightarrow (0, 1)$  be a non-decreasing map and let  $\eta > 0$ .*

*Suppose that*

$$\tau(p_i) \geq a_i \text{ and } \mu_{\tau \circ L}(O_a) \geq \Delta(a) \text{ for all } a \geq \eta \quad (\text{e 9.287})$$

*for all  $\tau \in T(A)$ .*

*Then, for any  $1 > r > 0$ , any  $1 > \delta > 0$ , any finite subset  $\mathcal{G} \subset C(X)$  and any finite subset  $\mathcal{H} \subset A$ , there exists a projection  $E \in A$ , a  $C^*$ -subalgebra  $B = \oplus_{j=1}^L C(X_j, M_{r(j)})$  with  $1_B = E$ , ( $X_j = [0, 1]$ , or  $X_j$  is a point), projections  $p'_i, p''_i$  with  $p'_i \in B$ , and contractive completely positive linear map  $L_1 : C(X) \rightarrow B$  with  $L_1(1_{C(X)})$  being a projection satisfying the following:*

$$\|Ea - aE\| < \delta \text{ for all } a \in \mathcal{H} \cup \{L(f) : f \in \mathcal{G}\}, \quad (\text{e 9.288})$$

$$\|p_i - (p'_i \oplus p''_i)\| < \delta, \quad i = 1, 2, \dots, n, \quad (\text{e 9.289})$$

$$\|L(f) - [EL(f)E + L_1(f)]\| < \delta \text{ for all } f \in \mathcal{G}, \quad (\text{e 9.290})$$

$$t_{j,x}(p'_i) \geq rd_i, \quad i = 1, 2, \dots, n \text{ and } \quad (\text{e 9.291})$$

$$\mu_{t_{j,x} \circ L_1}(O_a) \geq r\Delta(O_a) \text{ for all } a \geq \eta \quad (\text{e 9.292})$$

*for all  $x \in X_j$  and  $j = 1, 2, \dots, L$ . Moreover,*

$$\tau(1 - E) < \epsilon \text{ for all } \tau \in T(A).$$

*If  $L' : C(X) \rightarrow A$  is another  $\delta$ - $\mathcal{G}$ -multiplicative contractive completely positive linear map such that*

$$|\tau \circ L'(g) - \tau \circ L(g)| < \delta \text{ for all } g \in \mathcal{G}, \quad (\text{e 9.293})$$

we may further require that

$$\|EL'(f) - L'(f)E\| < \delta, \quad \|L'(f) - [EL'(f)E + L_1(f)]\| < \delta \text{ for all } f \in \mathcal{G}, \quad (\text{e 9.294})$$

$$|t_{j,x} \circ L_1(f) - t_{j,x} \circ L'_1(f)| < \epsilon \text{ and } \mu_{t_{j,x} \circ L'_1}(O_a) \geq r\Delta(a) \quad (\text{e 9.295})$$

for all  $x \in X_j$ ,  $j = 1, 2, \dots, L$ , for  $a \geq \eta$  and for all  $f \in \mathcal{G}$

*Proof.* There exists a sequence of projections  $E_n \in A$  and a sequence of  $C^*$ -subalgebra  $B_n = \oplus_{j=1}^{L_n} C(X_{j,n}, M_{r(j,n)})$  such that

$$\lim_{n \rightarrow \infty} \|E_n a - a E_n\| = 0 \text{ for all } a \in A. \quad (\text{e 9.296})$$

One then obtains a sequence of projections  $p'_{i,n} \in B_n$ ,  $p''_{i,n} \in (1 - E_n)A(1 - E_n)$  and a sequence of contractive completely positive linear maps  $\Phi_n : A \rightarrow B_n$  (see 2.3.9 of [22]) such that

$$\lim_{n \rightarrow \infty} \|p_i - (p'_{i,n} + p''_{i,n})\| = 0 \text{ and } \lim_{n \rightarrow \infty} \|a - [E_n a E_n + \Phi_n(a)]\| = 0 \quad (\text{e 9.297})$$

for all  $a \in A$ . Moreover,

$$\lim_{n \rightarrow \infty} \sup_{\tau \in T(A)} \{\tau(1 - e_n)\} = 0. \quad (\text{e 9.298})$$

Suppose that there exists a subsequence  $\{n_k\}$  such that

$$t_{j_{n_k}, x_k}(p'_{i,n_k}) < r d_i, \quad i = 1, 2, \dots, n. \quad (\text{e 9.299})$$

Define  $T_k(a) = t_{j_{n_k}, x_k}(\Phi_{n_k}(a))$  for  $a \in A$ . Let  $T$  be a limit point. Then  $T(1_A) = 1$ . So  $T$  is a state. It is easy to see that it is also a tracial state. Then

$$T(p_i) \leq r d_i, \quad i = 1, 2, \dots, n. \quad (\text{e 9.300})$$

A contradiction.

Suppose that there exists a subsequence  $\{n_k\}$  such that

$$\mu_{t_{j_{n_k}, x_k} \circ \Phi_{n_k} \circ L}(O_{a_k}) < r\Delta(a_k) \quad (\text{e 9.301})$$

for some  $1 > a_k \geq \eta$  and for all  $k$ . Again, use the above notation  $T$  for a limit of  $\{t_{j_{n_k}, x_k} \circ \Phi_{n_k}\}$ . Then  $T$  is a tracial state so that

$$\mu_{T \circ L}(O_a) \leq r\Delta(a) \quad (\text{e 9.302})$$

for some  $a \geq \eta$ . Another contradiction.

The first part of the lemma follows by choosing  $L_1$  to be  $\Phi_n \circ L$ ,  $p'_i$  to be  $p'_{1,n}$  and  $p''_i$  to be  $p''_{1,n}$  for some sufficiently large  $n$ .

The last part follows from a similar argument. □

**Lemma 9.9.** *Let  $A$  be a unital separable simple  $C^*$ -algebra with tracial rank no more than one. Suppose that  $p, q \in A$  are two projections such that*

$$\tau(p) \geq D \text{ and } \tau(q) \geq D \text{ for all } \tau \in T(A).$$

*Then, for any  $1 > r > 1$ , there are projections  $p_1 \leq p$  and  $q_1 \leq q$  such that*

$$[p_1] = [q_1] \text{ in } K_0(A) \text{ and } \tau(p_1) = \tau(q_1) \geq r \cdot D \quad (\text{e 9.303})$$

*for all  $\tau \in T(A)$ .*

*Proof.* Fix  $1 > r_1 > r > 0$ .

Similar argument as in 9.7 leads to the following: there are mutually orthogonal projections  $p'_0, p'_1$  and mutually orthogonal projections  $q'_0, q'_1$  such that

$$\|p_0 + p'_1 - p\| < 1/2, \quad \|q_0 + q'_1 - q\| < 1/2 \quad (\text{e 9.304})$$

and  $p'_1, q'_1 \in B = \oplus_{j=1}^L C(X_j, M_{r(j)})$ , where  $X_j = [0, 1]$ , or  $X_j$  is a single point,

$$t_{j,x}(p'_1) > r_1 D \quad \text{and} \quad t_{j,x}(q'_1) > r_1 D \quad (\text{e 9.305})$$

for  $x \in X_j$  and  $j = 1, 2, \dots, L$ . Moreover, as in 3.4 of [26],  $r(j) \geq \frac{2}{(r_1-r)D}$ . There is a projection  $p_{1,j} \in C(X_j, M_{r(j)})$  such that  $p_{1,j} \leq \pi_j(p'_1)$  and

$$r_1 D \geq t_{j,x}(p_{1,j}) > r D \quad (\text{e 9.306})$$

for  $x \in X_j$  and  $j = 1, 2, \dots, L$ , where  $\pi_j : B \rightarrow C(X_j, M_{r(j)})$  is a projection.

Since

$$t_{j,x}(p_{1,j}) \leq t_{j,x}(q'_1)$$

for all  $x \in X_j$ ,  $j = 1, 2, \dots, L$ . There exists a partial isometry  $v_j \in C(X_j, M_{r(j)})$  such that

$$v_j^* v_j = p_{1,j} \quad \text{and} \quad v_j v_j^* \leq \pi_j(q'_1),$$

$j = 1, 2, \dots, L$ .

Define  $p''_1 = \sum_{j=1}^L p_{1,j}$  and  $v = \sum_{j=1}^L v_j$ . Then

$$p''_1 \leq p_1, v^* v = p''_1 \quad \text{and} \quad v v^* \leq q'_1.$$

Moreover,

$$\tau(p''_1) \geq r D \quad \text{for all } \tau \in T(A).$$

By (e 9.304), there exists projection  $p_1 \leq p$  and a projection  $q_1 \leq q$  such that

$$[p_1] = [p''_1] = [v v^*] = [q_1]. \quad (\text{e 9.307})$$

Note that

$$\tau(p_1) = \tau(q_1) \geq r \cdot D \quad \text{for all } \tau \in T(A).$$

□

**Lemma 9.10.** (cf. Lemma 5.5 of [26]) *Let  $B$  be a unital separable amenable  $C^*$ -algebra and let  $A$  be a unital simple  $C^*$ -algebra with  $TR(A) \leq 1$ . For any  $\epsilon > 0$ , any finite subset  $\mathcal{F} \subset B$ , any  $\sigma > 0$ , any integer  $k \geq 1$ , and integer  $K \geq 1$  and any finite subset  $\mathcal{F}_1 \subset A$ . Suppose that  $\phi, \psi : B \rightarrow A$  are two unital positive linear maps. Then, there is a projection  $p \in A$ , a  $C^*$ -subalgebra  $C_0 = \oplus_{i=1}^{n_1} (C([0, 1], M_{d(i)}) \oplus \oplus_{j=1}^{n_2} M_{r(j)})$  with  $d(i), r(j) \geq K$  and a  $C^*$ -subalgebra  $C$  of  $A$  with  $C = M_k(C_0)$  and with  $1_C = p$  and unital positive linear maps  $\phi_0, \psi_0 : B \rightarrow C_0$  such that*

$$\|[\phi(f), p]\| < \epsilon, \quad \|[\psi(f), p]\| < \epsilon \quad \text{for all } f \in \mathcal{F}; \quad (\text{e 9.308})$$

$$\|[x, p]\| < \epsilon \quad \text{for all } x \in \mathcal{F}_1; \quad (\text{e 9.309})$$

$$\|\phi(f) - ((1-p)\phi(f)(1-p) \oplus \phi_0^{(k)}(f))\| < \epsilon, \quad (\text{e 9.310})$$

$$\|\psi(f) - ((1-p)\psi(f)(1-p) \oplus \psi_0^{(k)}(f))\| < \epsilon \quad \text{for all } f \in \mathcal{F} \quad \text{and} \quad (\text{e 9.311})$$

$$\tau(1-p) < \sigma \quad \text{for all } \tau \in T(A), \quad (\text{e 9.312})$$

where

$$\phi_0^{(k)}(f) = \text{diag}(\overbrace{\phi_0(f), \phi_0(f), \dots, \phi_0(f)}^k) \text{ and} \quad (\text{e 9.313})$$

$$\psi_0^{(k)}(f) = \text{diag}(\overbrace{\psi_0(f), \psi_0(f), \dots, \psi_0(f)}^k) \text{ for all } f \in B. \quad (\text{e 9.314})$$

*Proof.* Fix  $\epsilon_1 > 0$  and a finite subset  $\mathcal{G} \subset B$ . We assume  $\epsilon_1 < \epsilon/16$ . Choose an integer  $N$  such that  $\frac{k}{N} < \sigma/4$ .

Since  $A$  is tracially approximately divisible, there exists a projection  $q \in A$  and a finite dimensional  $C^*$ -subalgebra  $D = \oplus_{i=1}^n M_{R(i)}$  with  $R(i) \geq N$  and with  $1_D = p$  such that

$$\|[x, y]\| < \epsilon_1/8k \text{ for all } x \in \mathcal{F}_1 \cup \{\phi(f), \psi(f) : f \in \mathcal{G}\} \text{ and } y \in D \quad (\text{e 9.315})$$

with  $\|y\| \leq 1$  and  $\tau(1 - q) < \sigma/4$  for all  $\tau \in T(A)$ . Write  $R(i) = m_i k + s_i$ , where  $m_i > 0$  and  $s_i \geq 0$  are integers such that  $s_i < k$ ,  $i = 1, 2, \dots, n$ . Since  $R(i) \geq N$ ,

$$\frac{s_i}{R(i)} < \sigma/4, \quad i = 1, 2, \dots, n. \quad (\text{e 9.316})$$

Let  $\{e_{l,j}^{(i)}\}_{1 \leq l, j \leq R(i)}$  be a matrix unit for  $M_{R(i)}$ ,  $i = 1, 2, \dots, n$ . Choose  $e_i = \sum_{j=1}^{m_i k} e_{j,j}^{(i)}$ . Define  $D_1 = \sum_{i=1}^n e_i M_{R(i)} e_i$ . Then  $D_1 \cong M_k(D_0)$  and  $D_0 = \sum_{i=1}^n M_{m_i}$ .

Put  $p' = \sum_{i=1}^n e_i$ . Then, by (e 9.316), we estimate that

$$\tau(1 - p') = \tau(1 - q) + \tau(q - \sum_{i=1}^n e_i) < \sigma/4 + \sigma/2 = \sigma/2 \quad (\text{e 9.317})$$

for all  $\tau \in T(A)$ . We have that

$$\|[x, y]\| < \epsilon_1/8k \text{ for all } x \in \mathcal{F}_1 \cup \{\phi(f), \psi(f) : f \in \mathcal{G}\} \text{ and } y \in D_1 \text{ with } \|y\| \leq 1. \quad (\text{e 9.318})$$

Let  $E_1, E_2, \dots, E_k$  be mutually orthogonal and mutually equivalent projections in  $M_k(D_0)$  with  $E_1 = 1_{D_0}$ . Let  $w_i \in D_1$  be a unitary such that

$$w_i^* E_1 w_i = E_i, \quad i = 1, 2, \dots.$$

Since  $TR(E_1 A E_1) \leq 1$ , there exists a projection  $e_1 \in E_1 A E_1$  and a  $C^*$ -subalgebra  $C_0$  of  $E_1 A E_1$  with  $C_0 = \oplus_{i=1}^{n_1} (C([0, 1], M_{d(i)}) \oplus \oplus_{j=1}^{n_2} M_{r(j)})$  with  $d(i), r(j) \geq K$ ,  $1 \leq i \leq n_1$  and  $1 \leq j \leq n_2$ , and with  $1_{C_0} = q_1$  such that

$$\|[x, q_1]\| < \epsilon_1/16k \text{ for all } x \in \mathcal{F}_1 \cup \{p' \psi(f) p', p' \phi'_0(f) p' : f \in \mathcal{G}\}; \quad (\text{e 9.319})$$

$$\text{dist}(q_1 x q_1, C_0) < \epsilon_1/16k \text{ for all } x \in \mathcal{F}_1 \cup \{p' \psi(f) p', p' \phi'_0(f) p' : f \in \mathcal{G}\} \text{ and } \quad (\text{e 9.320})$$

$$t(E_1 - q_1) < \sigma/16k \text{ for all } t \in T(E_1 A E_1). \quad (\text{e 9.321})$$

It follows from 3.2 of [20] that there exists a unital contractive completely positive linear map  $\phi_0, \psi_0 : B \rightarrow C_0$  such that

$$\|q_1 \phi(f) q_1 - \phi_0(f)\| < \epsilon/16k \text{ and } \|q_1 \psi_0(f) q_1 - \psi_0(f)\| < \epsilon/16k \text{ for all } f \in \mathcal{F}, \quad (\text{e 9.322})$$

provided that  $\epsilon_1$  is small enough and  $\mathcal{G}$  is large enough. Put  $p = \sum_{i=1}^k w_i^* q_1 w_i$  and  $q_i = w_i^* q_1 w_i$ ,  $i = 1, 2, \dots, k$ . Then, by (e 9.317) and (e 9.321),

$$\tau(1 - p) < \sigma \text{ for all } \tau \in T(A). \quad (\text{e 9.323})$$

We estimate that

$$\sum_{i=1}^k w_i^* q_1 \phi(f) q_1 w_i = \sum_{i=1}^k q_i w_i^* q_1 \phi(f) q_1 w_i \quad (\text{e 9.324})$$

$$\approx_{\epsilon_1/8k} \sum_{i=1}^k q_i \phi(f) w_i^* q_1 w_i \quad (\text{by (e 9.319) and (e 9.315)}) \quad (\text{e 9.325})$$

$$= \sum_{i=1}^k q_i \phi(f) q_i \quad \text{on } \mathcal{F}. \quad (\text{e 9.326})$$

Thus, by (e 9.319), (e 9.326) and (e 9.322),

$$p \phi(f) p \approx_{\epsilon_1/4} \sum_{i=1}^k q_i \phi(f) q_i \quad (\text{e 9.327})$$

$$\approx_{\epsilon_1/8k} \sum_{i=1}^k w_i^* q_1 \phi(f) q_1 w_i \quad (\text{e 9.328})$$

$$\approx_{\epsilon/16k} \sum_{i=1}^k w_i^* \phi_0(f) w_i \quad \text{on } \mathcal{F}. \quad (\text{e 9.329})$$

Exactly the same argument shows that

$$p \psi(f) p \approx_{\epsilon/4} \sum_{i=1}^k w_i^* \psi_0(f) w_i \quad \text{on } \mathcal{F}. \quad (\text{e 9.330})$$

The lemma follows by combining together (e 9.329), (e 9.330), (e 9.318), (e 9.319) and (e 9.323).  $\square$

## 10 Approximate unitary equivalence

**Lemma 10.1.** *Let  $X$  be a connected finite CW complex and let  $Y = X \setminus \{\xi\}$ , where  $\xi \in X$  is a point. Let  $K_0(C(Y)) = G = \mathbb{Z}^k \oplus \text{Tor}(G)$  and  $K_0(C(X)) = \mathbb{Z} \oplus G$ . Fix  $\kappa \in \text{Hom}_\Lambda(\underline{K}(C_0(Y)), \underline{K}(\mathcal{K}))$ .*

*Put  $K = \max\{|\kappa(g_i)| : g_i = \overbrace{(0, \dots, 0)}^{i-1}, 1, 0, \dots, 0) \in \mathbb{Z}^k\}$ . Then, for any  $\delta > 0$  any finite subset  $\mathcal{G} \subset C(X)$  and any finite subset  $\mathcal{P} \subset \underline{K}(C_0(Y))$ , there exists an integer  $N(K) \geq 1$  (which depends on  $K$ ,  $\delta$ ,  $\mathcal{G}$  and  $\mathcal{P}$ , but not  $\kappa$ ) and a unital  $\delta$ - $\mathcal{G}$ -multiplicative contractive completely positive linear map  $L : C(X) \rightarrow M_{N(K)}$  such that*

$$[L|_{C_0(Y)}]|_{\mathcal{P}} = \kappa|_{\mathcal{P}}. \quad (\text{e 10.331})$$

(Note that the lemma includes the case that  $K = 0$ .)

*Proof.* Choose  $\delta_0 > 0$  and a finite subset  $\mathcal{G}_0 \subset C_0(Y)$  such that, for any pair of  $\delta_0$ - $\mathcal{G}_0$ -multiplicative contractive completely positive linear maps from  $C_0(Y)$  to any  $C^*$ -algebra,  $[L_i]|_{\mathcal{P}}$  is well-defined and

$$[L_1]|_{\mathcal{P}} = [L_2]|_{\mathcal{P}}, \quad (\text{e 10.332})$$

provided that

$$L_1 \approx_{\delta_0} L_2 \quad \text{on } \mathcal{G}_0.$$

It follows from 4.3 and 5.3 of [6] that there exists an asymptotic morphism  $\{\phi_t : t \in [1, \infty)\} : C_0(Y) \rightarrow \mathcal{K}$  such that

$$[\{\phi_t\}] = \kappa. \quad (\text{e 10.333})$$

Note that, for each  $t \in [1, \infty)$ ,  $\phi_t$  is a contractive completely positive linear map and

$$\lim_{t \rightarrow \infty} \|\phi_t(ab) - \phi_t(a)\phi_t(b)\| = 0$$

for all  $a, b \in C_0(Y)$ . Define  $\delta_1 = \min\{\delta_0/2, \delta/2\}$  and  $\mathcal{G}_1 = \mathcal{G}_0 \cup \mathcal{G}$ . It follows that, for sufficiently large  $t$ ,

$$[\phi_t]|_{\mathcal{P}} = \kappa|_{\mathcal{P}} \quad (\text{e 10.334})$$

and  $\phi_t$  is  $\delta_1$ - $\mathcal{G}_1$  multiplicative. Choose a projection  $E \in \mathcal{K}$  such that

$$\|E\phi_t(a) - \phi_t(a)E\| < \delta_1/4 \text{ for all } a \in \mathcal{G}_2, \quad (\text{e 10.335})$$

where  $\mathcal{G}_2 = \mathcal{G}_1 \cup \{ab : a, b \in \mathcal{G}_1\}$ . Define  $L : C(X) \rightarrow E\mathcal{K}E$  by  $L(f) = f(\xi)E + E\phi_t(f - f(\xi))E$  for  $f \in C(X)$ . It is easy to see that  $L$  is a  $\delta$ - $\mathcal{G}$ -multiplicative contractive completely positive linear map and

$$[L|_{C_0(Y)}]|_{\mathcal{P}} = \kappa|_{\mathcal{P}}. \quad (\text{e 10.336})$$

Define the rank of  $E$  to be  $N(\kappa)$ . Note that  $E\mathcal{K}E \cong M_{N(\kappa)}$ . Note that since  $K_i(C_0(Y))$  is finitely generated, by [7],

$$\text{Hom}_{\Lambda}(\underline{K}(C_0(Y)), \underline{K}(\mathcal{K})) = \text{Hom}_{\Lambda}(F_m \underline{K}(C_0(Y)), F_m \underline{K}(\mathcal{K}))$$

for some integer  $m \geq 1$ . Thus, when  $K$  is given, there are only finitely many different  $\kappa$  so that  $|\kappa(g_i)| \leq K$  for  $i = 1, 2, \dots, k$ . Thus such  $N(K)$  exists by taking the maximum of those  $N(\kappa)$ .  $\square$

**Lemma 10.2.** *Let  $X$  be a connected finite CW complex and let  $Y = X \setminus \{\xi\}$ , where  $\xi \in X$  is a point. Let  $K_0(C(Y)) = G = \mathbb{Z}^k \oplus \text{Tor}(G)$  and  $K_0(C(X)) = \mathbb{Z} \oplus G$ . For any  $\delta > 0$ , any finite subset  $\mathcal{G} \subset C(X)$  and any finite subset  $\mathcal{P} \subset \underline{K}(C_0(Y))$ , there exists an integer  $N(\delta, \mathcal{G}, \mathcal{P}) \geq 1$  satisfying the following:*

*Let  $\kappa \in \text{Hom}_{\Lambda}(\underline{K}(C_0(Y)), \underline{K}(\mathcal{K}))$  and let  $K = \max\{|\kappa(g_i)| : g_i = (\overbrace{0, \dots, 0}^{i-1}, 1, 0, \dots, 0) \in \mathbb{Z}^k\}$ . There exists an integer  $N(K) \geq 1$  and a unital  $\delta$ - $\mathcal{G}$ -multiplicative contractive completely positive linear map  $L : C(X) \rightarrow M_{N(K)}$  such that*

$$[L]|_{\mathcal{P}} = \kappa|_{\mathcal{P}} \text{ and } \frac{N(K)}{\max\{K, 1\}} \leq N(\delta, \mathcal{G}, \mathcal{P}). \quad (\text{e 10.337})$$

*Proof.* Fix  $\delta$ ,  $\mathcal{P}$  and  $\mathcal{G}$ . Let  $N(0)$  and  $N(1)$  be in 10.1 corresponding to the case that  $K = 0$  and  $K = 1$ . Define

$$N(\delta, \mathcal{G}, \mathcal{P}) = kN(1) + N(0).$$

Fix  $\kappa \in \text{Hom}_{\Lambda}(\underline{K}(C_0(Y)), \underline{K}(\mathcal{K}))$ . Suppose that  $\kappa(g_i) = m_i$ ,  $i = 1, 2, \dots, k$ . For each  $i$ , ( $i = 0, 1, 2, \dots, k$ ) there is  $\kappa_i \in \text{Hom}_{\Lambda}(\underline{K}(C_0(Y)), \underline{K}(\mathcal{K}))$  such that

$$\kappa_0(g_i) = 0, \quad i = 1, 2, \dots, k, \quad (\text{e 10.338})$$

$$\kappa_i(g_j) = 0, \quad \text{if } m_i = 0, \quad j = 1, 2, \dots, k \quad (\text{e 10.339})$$

$$\kappa_i(g_i) = \text{sign}(m_i) \cdot 1 \text{ (in } \mathbb{Z}) \text{ and } \kappa_i(g_j) = 0 \text{ if } j \neq i, \quad (\text{e 10.340})$$

$$\text{if } m_i \neq 0, \quad i = 1, 2, \dots, k \text{ and} \quad (\text{e 10.341})$$

$$\kappa_0 + \sum_{i=1}^k m_i \kappa_i = \kappa. \quad (\text{e 10.342})$$

By 10.1, there exists a unital  $\delta\mathcal{G}$ -multiplicative contractive completely positive linear map  $L_i : C(X) \rightarrow M_{N(1)}$  such that

$$[L_i|_{C_0(Y)}]|_{\mathcal{P}} = \kappa_i|_{\mathcal{P}}, \quad i = 0, 1, 2, \dots, k. \quad (\text{e 10.343})$$

Put  $N = N(0) + \sum_{i=1}^k |m_i|N(1)$ . Define  $L : C(X) \rightarrow M_N$  by

$$L(f) = L_0(f) \oplus \oplus_{i=1}^k \bar{L}_i(f), \quad (\text{e 10.344})$$

for all  $f \in C(X)$ , where

$$\bar{L}_i(f) = \text{diag}(\overbrace{L_i(f), L_i(f), \dots, L_i(f)}^{|m_i|}), \quad i = 1, 2, \dots, k. \quad (\text{e 10.345})$$

One estimates that

$$\frac{N}{\max\{K, 1\}} = \frac{N(0) + \sum_{i=1}^k |m_i|N(1)}{\max\{K, 1\}} \leq N(0) + kN(1) = N(\delta, \mathcal{G}, \mathcal{P}). \quad (\text{e 10.346})$$

□

**Lemma 10.3.** *Let  $X$  be a connected finite CW complex with  $K_0(C(X)) = \mathbb{Z} \oplus G$ , where  $G = \mathbb{Z}^k \oplus \text{Tor}(G) = K_0(C_0(Y))$  and  $Y = X \setminus \{\xi\}$  for some point  $\xi \in X$ . For any  $\sigma > 0$ , there exists  $\delta > 0$  and a finite subset  $\mathcal{G} \subset C(X)$  satisfying the following:*

*For any unital separable  $C^*$ -algebra  $A$  with  $T(A) \neq \emptyset$  and any unital  $\delta\mathcal{G}$ -multiplicative contractive completely positive linear map  $L : C(X) \rightarrow A$ , one has*

$$|\tau \circ [L](g_i)| < \sigma \text{ for all } \tau \in T(A), \quad (\text{e 10.347})$$

where  $g_i = (\overbrace{0, \dots, 0}^{i-1}, 1, 0, \dots, 0) \in \mathbb{Z}^k$  and  $\tau$  is the state on  $K_0(C(X))$  induced by the tracial state  $\tau$ .

*Proof.* Suppose that the lemma is false.

Then there exists a sequence of unital separable  $C^*$ -algebras  $A_n$  and a sequence of  $\delta_n\mathcal{G}_n$ -multiplicative contractive completely positive linear maps  $L_n : C(X) \rightarrow A_n$ , where  $\delta_n \downarrow 0$  and  $\mathcal{G}_n$  is a sequence of finite subsets such that  $\mathcal{G}_n \subset \mathcal{G}_{n+1}$  and  $\cup_{n=1}^\infty \mathcal{G}_n$  is dense in  $C(X)$  and there exists  $\tau_n \in T(A_n)$  such that

$$|\tau_n \circ [L_n](g_i)| \geq \sigma/2 \quad (\text{e 10.348})$$

for some  $i \in \{1, 2, \dots, k\}$ .

Let  $B = \prod_{n=1}^\infty A_n$ . Define  $t_n(\{a_n\}) = \tau_n(a_n)$ . Then  $t_n$  is a tracial state of  $B$ . Let  $T$  be a limit point of  $\{t_n\}$ . One obtains a subsequence  $\{n_k\}$  such that

$$T(\{a_n\}) = \lim_{k \rightarrow \infty} \tau_{n_k}(a_{n_k}) \quad (\text{e 10.349})$$

for any  $\{a_n\} \in B$ . Note for any  $a \in \oplus_{n=1}^\infty A_n \subset B$ ,  $T(a) = 0$ . It follows that  $T$  defines a tracial state  $\bar{T}$  on  $B / \oplus_{n=1}^\infty A_n$ . Let  $\Pi : B \rightarrow B / \oplus_{n=1}^\infty A_n$  be the quotient map. Define  $L : C(X) \rightarrow B$  by  $L(f) = \{L_n(f)\}$ . Put  $\phi = \Pi \circ L$ . Then  $\phi$  is a unital homomorphism. Therefore

$$\bar{T} \circ \phi_{*0}(g_i) = 0. \quad (\text{e 10.350})$$

It follows that there is a subsequence  $\{n'_k\} \subset \{n_k\}$  such that

$$\lim \tau_{n'_k} \circ [L_{n'_k}](g_i) = 0. \quad (\text{e 10.351})$$

But this contradicts with (e 10.348).

□



**Lemma 10.4.** *Let  $C(X)$  be a connected finite CW complex and  $\mathcal{P} \subset \underline{K}(C(X))$ . There exists  $\delta > 0$  and there exists a finite subset  $\mathcal{G} \subset C(X)$  satisfying the following: for any unital  $C^*$ -algebra  $A$ , and any unital  $\delta$ - $\mathcal{G}$ -multiplicative contractive completely positive linear map  $L : C(X) \rightarrow A$ , there exists  $\kappa \in \text{Hom}_\Lambda(\underline{K}(C(X)), \underline{K}(A))$  such that*

$$[L]|_{\mathcal{P}} = \kappa|_{\mathcal{P}}. \quad (\text{e 10.352})$$

This is known (see Prop. 2.4 of [29]).

**Lemma 10.5.** *Let  $X \in \mathbf{X}$  be a finite simplicial complex. Let  $\epsilon > 0$ , let  $\epsilon_1 > 0$ , let  $\eta_0 > 0$ , let  $\mathcal{F} \subset C(X)$  be a finite subset, let  $N \geq 1$  and  $K \geq 1$  be positive integers and let  $\Delta : (0, 1) \rightarrow (0, 1)$  be a non-decreasing map. There exist  $\eta > 0$ ,  $\delta > 0$ , a finite subset  $\mathcal{G}$  and a finite subset  $\mathcal{P} \subset \underline{K}(C(X))$  satisfying the following:*

*Suppose that  $A$  is a unital separable simple  $C^*$ -algebra with tracial rank no more than one and  $\phi, \psi : C(X) \rightarrow A$  are two unital  $\delta$ - $\mathcal{G}$ -multiplicative contractive completely positive linear maps such that*

$$\mu_{\tau \circ \phi}(O_a) \geq \Delta(a) \text{ for all } a \geq \eta, \quad (\text{e 10.353})$$

$$|\tau \circ \phi(g) - \tau \circ \psi(g)| < \delta \text{ for all } g \in \mathcal{G} \quad (\text{e 10.354})$$

*for all  $\tau \in T(A)$  and*

$$[\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}}. \quad (\text{e 10.355})$$

*Then, for any  $\epsilon_0 > 0$ , there are four mutually orthogonal projections  $P_0, P_1, P_2$  and  $P_3$  with  $P_0 + P_1 + P_2 + P_3 = 1_A$ , there is a unital  $C^*$ -subalgebra  $B_1 \subset (P_1 + P_2 + P_3)A(P_1 + P_2 + P_3)$  with  $1_B = P_1 + P_2 + P_3$ , where  $B_1$  has the form  $B_1 = \bigoplus_{j=1}^s C(X_j, M_{r(j)})$  with  $P_1, P_2, P_3 \in B_1$ , where  $X_j = [0, 1]$ , or  $X_j$  is a point, there are unital homomorphisms  $\phi_1, \psi_1 : C(X) \rightarrow B$ , where  $B = P_3 B_1 P_3$ , there exists a finite dimensional  $C^*$ -subalgebra  $C_0 \subset P_1 B P_1$  with  $1_{C_0} = P_1$  and there exists a unital  $\epsilon$ - $\mathcal{F}$ -multiplicative contractive completely positive linear map  $\phi_2 : C(X) \rightarrow C_0$  and mutually orthogonal projections  $p_1, p_2, \dots, p_m \in P_2 B_1 P_2$  and a unitary  $u \in A$  such that*

$$\|\phi(f) - [P_0 \phi(f) P_0 + \phi_2(f) + \sum_{i=1}^m f(x_i) p_i + \phi_1(f)]\| < \epsilon/2 \text{ and} \quad (\text{e 10.356})$$

$$\|\text{ad } u \circ \psi(f) - [P_0 (\text{ad } u \circ \psi(f)) P_0 + \phi_2(f) + \sum_{i=1}^m f(x_i) p_i + \psi_1(f)]\| < \epsilon/2 \quad (\text{e 10.357})$$

*for all  $f \in \mathcal{F}$ , where  $\{x_1, x_2, \dots, x_m\}$  is  $\epsilon_1$ -dense in  $X$  and  $P_2 = \sum_{i=1}^m p_i$ ,*

$$N\tau(P_0 + P_1) < \tau(p_i) \quad K t_{j,x}(P_1 + P_2) \leq t_{j,x}(P_3) \quad (\text{e 10.358})$$

$$\mu_{T \circ \phi_1}(O_a) \geq \Delta(a)/4, \quad \mu_{T \circ \psi_1}(O_a) \geq \Delta(a)/4 \text{ for all } a \geq \eta_0 \quad (\text{e 10.359})$$

$$|T \circ \psi_1(f) - T \circ \phi_1(f)| < \epsilon \text{ for all } f \in \mathcal{F}, \quad (\text{e 10.360})$$

*for all  $\tau \in T(A)$ ,  $i = 1, 2, \dots, m$ , for all  $x \in X_j$ ,  $j = 1, 2, \dots, s$  and for all  $T \in T(B)$ . Moreover, for any finite subset  $\mathcal{H} \subset A$ , one may require that*

$$\|a P_0 - P_0 a\| < \epsilon_0 \text{ and } (1 - P_0) a (1 - P_0) \in_\epsilon B_1 \text{ for all } a \in \mathcal{H}. \quad (\text{e 10.361})$$

*Proof.* Without loss of generality, we may assume that  $X$  is connected. There is an integer  $k' \geq 1$  such that any torsion element in  $K_i(C(X))$  has order smaller than  $k'$ . Put  $k_0 = (k')!$ .

Let  $\epsilon > 0$ ,  $\epsilon_1 > 0$ ,  $\mathcal{F} \subset C(X)$ ,  $N$  and  $K$  are given. Let  $\epsilon > \epsilon_2 > 0$  and  $\mathcal{F}_0 \supset \mathcal{F}$  satisfying the following: if  $L_1, L_2 : C(X) \rightarrow B$  are two unital  $\epsilon_2$ - $\mathcal{F}_0$  multiplicative contractive completely positive linear maps (to any unital  $C^*$ -algebra  $B$  with  $T(B) \neq \emptyset$ ) such that

$$\mu_{T \circ L_1}(O_a) \geq \Delta(a)/2m \text{ for all } a \geq \eta_0 \text{ and } |T \circ L_1(f) - T \circ L_2(f)| < \epsilon_2 \quad (\text{e 10.362})$$

for all  $f \in \mathcal{G}_0$ , then

$$\mu_{T \circ L_2}(O_a) \geq \Delta(a)/4m \text{ for all } a \geq \eta_0, \quad (\text{e 10.363})$$

$m = 1, 2, \dots, k_0$ . Define  $\Delta_1(a) = \Delta(a)/4$  for  $a \in (0, 1)$ . Let  $\eta_1 > 0$  (in place of  $\eta$ ),  $\delta_1 > 0$  (in place of  $\delta$ ),  $\mathcal{G}_1 \subset C(X)$  (in place of  $\mathcal{G}$ ) be a finite subset and  $\mathcal{P}_1 \subset \underline{K}(C(X))$  (in place of  $\mathcal{P}$ ) be a finite subset required by 8.5 for  $\epsilon_2/16$  (in place of  $\epsilon$ ),  $\mathcal{F}_0$  (in place of  $\mathcal{F}$ ) and for both  $\Delta_1$  and  $\Delta_1/k_0$  (in place of  $\Delta$ ) above. We may assume that  $\delta_1 < \epsilon/64$ .

By choosing smaller  $\delta_1$  and large  $\mathcal{G}_1$ , we may assume that, if  $L_1, L_2 : C(X) \rightarrow B$  are two contractive completely positive linear maps (for any unital  $C^*$ -algebra  $B$  with  $T(B) \neq \emptyset$ ) and

$$\mu_{\tau \circ L_1}(O_a) \geq \Delta(a)/m \text{ for all } a \geq \eta_1$$

for all  $\tau \in T(B)$ , then

$$\mu_{\tau \circ L_2}(O_a) \geq 3\Delta(a)/4m \text{ for all } a \geq \eta_1 \quad (\text{e 10.364})$$

for all  $\tau \in T(B)$  and  $m = 1, 2, \dots, k_0$ , whenever

$$L_1 \approx_{\delta_1} L_2 \text{ on } \mathcal{G}_1. \quad (\text{e 10.365})$$

Let  $\xi \in X$  be a point in  $X$  and let  $Y = X \setminus \{\xi\}$ . Write  $K_0(C(X)) = \mathbb{Z} \oplus G$ , where  $\mathbb{Z}$  is given by the rank and  $G = \mathbb{Z}^k \oplus \text{Tor}(G) = K_0(C_0(Y))$ . Put  $g_i = (\overbrace{0, 0, \dots, 0}^{i-1}, 1, 0, \dots, 0)$ ,  $i = 1, 2, \dots, k$ . We may assume that  $g_i \in \mathcal{P}_1$ . In fact, since  $K_i(C(X))$  is finitely generated ( $i = 0, 1$ ), we may assume that  $[L]$  well defines an element in  $KK(C(X), A)$  for any  $\delta_1$ - $\mathcal{G}_1$ -multiplicative contractive completely positive linear map  $L : C(X) \rightarrow A$  (for any unital  $C^*$ -algebra  $A$ ) (see 10.4). For convenience, without loss of generality, we may further assume that,

$$[L_1]|_{\mathcal{P}_1} = [L_2]|_{\mathcal{P}_1} \quad (\text{e 10.366})$$

for any pair of  $\delta_1$ - $\mathcal{G}_1$  multiplicative contractive completely positive linear maps for which

$$L_1 \approx_{\delta_1} L_2 \text{ on } \mathcal{G}_1.$$

We may also assume that  $\mathcal{F} \subset \mathcal{G}_1$ . Let  $\eta_2 > 0$  (in place of  $\eta$ ) required by 4.1 such that

$$|f(x) - f(x')| < \delta_1/32 \text{ for all } f \in \mathcal{G}_1 \quad (\text{e 10.367})$$

if  $\text{dist}(x, x') < \eta_2$ . We may assume that  $\eta_2 < \epsilon_1/2$ .

Let  $s \geq 1$  for which there exists an  $\eta_2/2$ -dense subset  $\{x_1, x_2, \dots, x_m\}$  of  $X$  such that  $O_i \cap O_j = \emptyset$  ( $i \neq j$ ), where

$$O_i = \{x : x \in X : \text{dist}(x, x_i) < \eta_2/2s\}.$$

We may assume that  $\eta_2 < \eta_1/2$ . Let  $\sigma_1 = \frac{1}{k_0(2+m)\eta_2}$ . Let  $\delta_2 > 0$ , let  $\mathcal{G}_2 \subset C(X)$  and  $\mathcal{P}_2 \subset \underline{K}(C(X))$  be finite subsets required by 4.1 for the above  $\delta_1/2$  (in place of  $\epsilon$ ),  $\mathcal{G}_1$  (in place of  $\mathcal{F}$ ) and  $\eta_2$  (in place of  $\eta$ ),  $\sigma_1$  (in place of  $\sigma$ ) and  $s$  above.

For convenience, without loss of generality, we may assume that  $\delta_2 < \delta_1/16$  and  $\mathcal{G}_2 \supset \mathcal{F} \cup \mathcal{G}_2$  and  $\mathcal{P}_2 \supset \mathcal{P}_1$ . Without loss of generality, we may also assume that  $\mathcal{G}_2$  is in the unit ball of  $C(X)$ .

Let  $\eta = \eta_2/s$ . Let  $\delta_3 > 0$  and  $\mathcal{G}_3 \subset C(X)$  be a finite subset be as required by 9.6 for  $\delta_1/16$  ( in place of  $\epsilon$ ),  $\mathcal{G}_1$  ( in place of  $\mathcal{F}$ ) and  $\eta$  above. Choose integer  $N_1 \geq 64m$  such that

$$\frac{1}{N_1} < \delta_2/4. \quad (\text{e } 10.368)$$

Let  $N(\delta_2/2, \mathcal{G}_2, \mathcal{P}_2)$  be as in 10.2 and put

$$\sigma_2 = \frac{\Delta(\eta_2/2s)}{32N_1(K+2)(N+2)(1+N(\delta_2/2, \mathcal{G}_2, \mathcal{P}_2))}$$

Without loss generality, we may assume that  $N(\delta_2/2, \mathcal{G}_2, \mathcal{P}_2) \geq 2$ . We may assume that  $\mathcal{G}_3 \supset \mathcal{G}_2$  and  $\delta_3 < \delta_2/2$ . By 10.3, there exists  $\delta_4 > 0$  and a finite subset  $\mathcal{G}_4 \subset C(X)$  such that for any unital  $C^*$ -algebra  $B$  with  $T(B) \neq \emptyset$  and for any unital  $\delta_4$ - $\mathcal{G}_4$ -multiplicative contractive completely positive linear map  $L : C(X) \rightarrow B$ ,

$$\tau \circ [L](g_i) < \sigma_2/k_0. \quad (\text{e } 10.369)$$

Let  $\delta = \min\{\delta_4/2, \delta_3/2\}$ ,  $\mathcal{G} = \mathcal{G}_4 \cup \mathcal{G}_3$  and  $\mathcal{P} = \mathcal{P}_2 \cup \mathcal{P}_1$ . Let  $\mathcal{P}' = \mathcal{P} \cap \underline{K}(C_0(Y))$ . Suppose that  $\phi$  and  $\psi : C(X) \rightarrow A$  are two unital  $\delta$ - $\mathcal{G}$ -multiplicative contractive completely positive linear maps, where  $A$  is a unital separable simple  $C^*$ -algebra with tracial rank one or zero, satisfy the assumptions of the lemma for the above chosen  $\delta$ ,  $\mathcal{G}$  and  $\mathcal{P}$ .

In particular, we may assume that  $[\phi]$  and  $[\psi]$  define the same element in  $KK(C(X), A)$ , since  $K_i(C(X))$  is finitely generated.

It follows from 9.6 that there exist mutually orthogonal projections  $p'_1, p'_2, \dots, p'_m \in A$  and mutually orthogonal projections  $p''_1, p''_2, \dots, p''_m \in A$  such that

$$\|\phi(g) - [P'(\phi(g))P' + \sum_{j=1}^m f(x_j)p'_j]\| < \delta_1/16k_0 \text{ and} \quad (\text{e } 10.370)$$

$$\|\psi(g) - [P''(\psi(g))P'' + \sum_{j=1}^m f(x_j)p''_j]\| < \delta_1/16k_0 \quad (\text{e } 10.371)$$

for all  $g \in \mathcal{G}_1$ , where  $P' = 1 - \sum_{j=1}^m p'_j$  and  $P'' = \sum_{j=1}^m p''_j$ . Moreover

$$\tau(p'_i) \geq (1 - \frac{1}{100k_0})\Delta(\eta) \text{ and } \tau(p''_i) \geq (1 - \frac{1}{100k_0})\Delta(\eta) \quad (\text{e } 10.372)$$

for all  $\tau \in T(A)$ ,  $i = 1, 2, \dots, m$ . By applying 9.5, and by replacing  $p'_i$  by one of its subprojection and  $p''_i$  by one of its subprojection, respectively, we replace (e 10.372) by the following:

$$\tau(p'_i) \geq (1/2)(1 - \frac{1}{100k_0})\Delta(\eta), \quad \sum_{i=1}^m \tau(p'_i) < 1/2, \quad (\text{e } 10.373)$$

$$\tau(p''_i) \geq (1/2)(1 - \frac{1}{100k_0})\Delta(\eta) \text{ and } \sum_{i=1}^m \tau(p''_i) < 1/2 \quad (\text{e } 10.374)$$

for all  $\tau \in T(A)$ ,  $i = 1, 2, \dots, m$ . By 9.9, there are projections  $q'_i \leq p'_i$  and  $q''_i \leq p''_i$  such that

$$\tau(q'_i) \geq (\frac{1}{2} - \frac{1}{100k_0})\Delta(\eta) \text{ and } [q''_i] = [q'_i] \text{ (in } K_0(A)), \quad (\text{e } 10.375)$$

$i = 1, 2, \dots, m$ . There is a unitary  $u \in A$  such that

$$u^* q_i'' u = q_i', \quad i = 1, 2, \dots, m. \quad (\text{e 10.376})$$

Therefore, we have

$$\|\phi(f) - [Q'\phi(f)Q' + \sum_{j=1}^m f(x_j)q_j']\| < \delta_1/16k_0 \quad \text{and} \quad (\text{e 10.377})$$

$$\|\text{ad } u \circ \psi(f) - [Q'\text{ad } u \circ \psi(f)Q' + \sum_{j=1}^m f(x_j)q_j']\| < \delta_1/16k_0 \quad (\text{e 10.378})$$

for all  $f \in \mathcal{G}_1$ . We also have that

$$\sum_{i=1}^m \tau(q_i') < 1/2 \quad \text{for all } \tau \in T(A). \quad (\text{e 10.379})$$

Note that

$$[Q'\phi Q'|_{C_0(Y)}] = [\phi|_{C_0(Y)}] = [\psi|_{C_0(Y)}] = [Q'(\text{ad } u \circ \psi)Q'|_{C_0(Y)}]. \quad (\text{e 10.380})$$

Let  $\mathcal{H} \subset C(X)$  be given. Since  $A$  has tracial rank no more than one, by applying 9.10, we obtains a projection  $E \in A$  and a unital  $C^*$ -subalgebra  $B_1 = \oplus_{j=1}^L C(X_j, M_{r(j)})$  ( $X_j = [0, 1]$  or  $X_j$  is a single point) with  $1_{B_1} = 1 - E$  satisfying the following:

$$\|\phi(f) - [E(\phi(f))E + \sum_{i=1}^m f(x_i)e_i + \phi_1'(f)]\| < \delta_1/8k_0 \quad (\text{e 10.381})$$

$$\|\text{ad } u \circ \psi(f) - [E(\text{ad } u \circ \psi(f))E + \sum_{i=1}^m f(x_i)e_i + \psi_1'(f)]\| < \delta_1/8k_0 \quad (\text{e 10.382})$$

for all  $f \in \mathcal{G}_1$  and

$$\|Ea - aE\| < \min\{\epsilon_0/2, \delta_1/8k_0\} \quad \text{and} \quad (1 - E)a(1 - E) \in_{\min\{\epsilon_0/2, \delta_1/8k_0\}} B_1 \quad (\text{e 10.383})$$

for all  $a \in \mathcal{H} \cup \mathcal{G}_1$ , where  $\sum_{i=1}^m e_i = E_1 \leq 1 - E$  and  $\phi_1', \psi_1' : C(X) \rightarrow B_2 = (1 - E - E_1)B_1(1 - E - E_1)$  is a  $\delta_1/4k_0$ - $\mathcal{G}_1$ -multiplicative contractive completely positive linear map and  $e_i \in B_1$ ,  $i = 1, 2, \dots, m$ . We may also assume that

$$r(j) > \frac{16}{\delta_2}, \quad j = 1, 2, \dots, L \quad (\text{e 10.384})$$

(see 3.3 of [26]). Moreover

$$\tau(E) < \sigma_2/4k_0 \quad \text{and} \quad \tau(e_i) > \left(\frac{1}{2} - \frac{1}{50k_0}\right)\Delta(\eta) \quad \text{for all } \tau \in T(A) \quad (\text{e 10.385})$$

Furthermore, by applying 9.10, we may assume that  $\phi_1'$  and  $\psi_1'$  have the form

$$\phi_1'(f) = \text{diag}(\overbrace{\phi_{1,0}'(f), \phi_{1,0}'(f), \dots, \phi_{1,0}'(f)}^{k_0}) \quad \text{and} \quad (\text{e 10.386})$$

$$\psi_1'(f) = \text{diag}(\overbrace{\psi_{1,0}'(f), \psi_{1,0}'(f), \dots, \psi_{1,0}'(f)}^{k_0}) \quad (\text{e 10.387})$$

for all  $f \in C(X)$ .

Since  $\mathcal{H}$  above is arbitrarily given, we may choose  $\mathcal{H}$  sufficiently large so that

$$[E\phi E]|_{\mathcal{P}} = [E(\text{ad } u \circ \psi)E]|_{\mathcal{P}} \text{ and } [\phi'_1]|_{\mathcal{P}} = [\psi'_1]|_{\mathcal{P}}. \quad (\text{e 10.388})$$

In particular, we may assume that (by (e 10.385) and (e 10.369))

$$\tau([\phi'_1](g_i)) < \sigma_2 + \sigma_2/16 \text{ and } \tau([\psi'_1](g_i)) < \sigma_2 + \sigma_2/16 \text{ for all } \tau \in T(A). \quad (\text{e 10.389})$$

Denote by  $\Phi, \Psi : C(X) \rightarrow B_1$  the maps defined by, for all  $f \in C(X)$ ,

$$\Phi(f) = \sum_{i=1}^m f(x_i)e_i + \phi'_1(f) \text{ and } \Psi(f) = \sum_{i=1}^m f(x_i)e_i + \psi'_1(f). \quad (\text{e 10.390})$$

Denote by  $\pi_j : B_1 \rightarrow C(X_j, M_{\bar{r}(j)})$  the projection. By 9.8, we may assume that

$$\mu_{t_{j,x} \circ \pi_j \circ \Phi}(O_a) \geq \frac{63\Delta(a)}{64} \text{ and } \mu_{t_{j,x} \circ \pi_j \circ \Psi}(O_a) \geq \frac{63\Delta(a)}{64} \text{ for all } a \geq \eta, \quad (\text{e 10.391})$$

where  $t_{j,x}$  is the standard normalized trace evaluated at  $x \in X_j$ ,  $j = 1, 2, \dots, L$ . We may also assume that

$$|t_{j,x} \circ \Phi(f) - t_{j,x} \circ \Psi(f)| < \delta_1/4 \text{ for all } f \in \mathcal{F}. \quad (\text{e 10.392})$$

Also by 9.8, we may also assume that

$$t_{j,x}(\pi_j(e_i)) > \frac{3\Delta(\eta)}{8}, \quad i = 1, 2, \dots, m, \quad (\text{e 10.393})$$

for  $x \in X_j$  and  $j = 1, 2, \dots, L$ . Moreover,

$$\sum_{i=1}^m t_{j,x}(\pi_j(e_i)) < 1/2. \quad (\text{e 10.394})$$

By 9.7, we may further assume that

$$t_{j,x}([\phi'_1](g_i)) < \frac{\sigma_2}{1 - \sigma_2/4} < \frac{4\sigma_2}{3}, \quad (\text{e 10.395})$$

$j = 1, 2, \dots, L$ . Note that every projection in  $C(X_j, M_{r(j)})$  is unitarily equivalent to a constant projection,  $j = 1, 2, \dots, L$ . Choose a rank one projection  $e_{11}^{(j)}$  in  $C(X_j, M_{r(j)})$ . Put

$$K_j = r(j) \max_i \{t_{j,x}([\phi'_1](g_i))\}, \quad j = 1, 2, \dots, L.$$

It follows from 10.2 that there is a  $\delta_2/2$ - $\mathcal{G}_2$ -multiplicative contractive completely positive linear map  $\lambda_i \cdot \bar{\lambda}_i : C(X) \rightarrow M_{R(j)}$  such that

$$[\lambda_j|_{C_0(Y)}]|_{\mathcal{P}'} = [\pi_j \circ \phi'_1|_{C_0(Y)}]|_{\mathcal{P}'} \text{ and } [\bar{\lambda}_j|_{C_0(Y)}]|_{\mathcal{P}'} = -[\pi_j \circ \phi'_1|_{C_0(Y)}]|_{\mathcal{P}'}, \quad (\text{e 10.396})$$

where  $\pi_j : B_1 \rightarrow C(X_j, M_{r(j)})$  is the projection and  $R(j) \leq K_j N(\delta_2/2, \mathcal{G}_2, \mathcal{P}_2)$ .

To simplify notation, we now identify  $M_{R(j)}$  with a  $C^*$ -subalgebra  $C_j$  of  $\pi_j(B_1)$  with constant matrices (with rank one projection  $e_{11}^{(j)}$  given earlier). We compute that

$$t_{j,x}(1_{C_j}) = R(j)t_{j,x}(e_{11}^{(j)}) \quad (\text{e 10.397})$$

$$\leq K_j N(\delta_2, \mathcal{G}_2, \mathcal{P}_2) \cdot t_{j,x}(e_{11}^{(j)}) \quad (\text{see (e 10.395)}) \quad (\text{e 10.398})$$

$$< (4/3)\sigma_2 \cdot N(\delta_2, \mathcal{G}_2, \mathcal{P}_2) = \frac{\Delta(\eta/2s)}{24N_1(N+2)(K+2)}. \quad (\text{e 10.399})$$

for all  $x \in X_j$ ,  $j = 1, 2, \dots, L$ . Since (by (e 10.393))

$$(N_1 + 2)t_{j,x}(1_{C_j}) < t_{j,x}(\pi_j(e_i)), \quad i = 1, 2, \dots, m,$$

there exists a projection  $e'_{j,i} \leq \pi_j(e_i)$  and a projection  $e''_{j,i} \leq \pi_j(e_i) - e_{j,i'}$  so that

$$[e'_{j,i}] = N_1[1_{C_j}] \quad \text{and} \quad [e''_{j,i}] = 2[1_{C_j}] \quad \text{in} \quad K_0(C(X_j, M_{r(j)})). \quad (\text{e 10.400})$$

Put  $e'_i = \sum_{j=1}^L e'_{j,i}$  and  $e''_i = \sum_{j=1}^L e''_{j,i}$ ,  $i = 1, 2, \dots, m$ . Define  $\lambda : C(X) \rightarrow M_2(B_1)$  by

$$\lambda(f) = \sum_{j=1}^L \lambda_j(f) \oplus \bar{\lambda}_j(f) \oplus \sum_{i=1}^m f(x_i)e'_i + \sum_{i=2}^m f(x_i)e''_i \quad (\text{e 10.401})$$

for all  $f \in C(X)$ . In fact, there exists a finite dimensional  $C^*$ -subalgebra  $C' \subset B_1$  such that  $\lambda$  maps  $C(X)$  into  $M_2(C')$  unittally.

Consider homomorphism  $h : C(X) \rightarrow M_2(B_1)$  defined by

$$h(f) = \sum_{i=1}^m f(x_i)(e'_i + e''_i) \quad \text{for all } f \in C(X). \quad (\text{e 10.402})$$

Define  $E' = \sum_{i=1}^m e'_i + e''_i$ . There is a unitary  $w \in M_2(B_1)$  such that

$$w^* \lambda(1_{C(X)}) w = E'.$$

Moreover, by (e 10.400), we can choose  $w$  so that  $\text{ad } w^* \circ h$  maps  $C(X)$  into  $M_2(C')$ . Let  $C = w^* C' w$ . So, in particular,  $C$  is of finite dimension. Note that  $C \subset E' B_1 E'$ .

We compute that

$$[\text{ad } w \circ \lambda]|_{\mathcal{P}} = [h]|_{\mathcal{P}}, \quad (\text{e 10.403})$$

$$\mu_{t \circ h}(O_{\eta_2/2s}(x_i)) \geq \frac{1}{2+m} \geq \sigma_1 \cdot \eta_2 \quad (\text{e 10.404})$$

for all  $t \in T(E' C E')$  and

$$|t(h(f)) - t(\lambda(f))| < 1/(1+m)N_1 < \delta_2 \quad \text{for all } f \in \mathcal{G}_2 \quad (\text{e 10.405})$$

and for all  $t \in T(E' C E')$ . By the choices of  $\delta_2$  and  $\mathcal{G}_2$  and applying 4.1, we obtain a unitary  $w_1 \in E' C E' \subset B_1$  such that

$$\text{ad } w_1 \circ \text{ad } w \circ \lambda \approx_{\delta_1/2} h \quad \text{on } \mathcal{G}_1. \quad (\text{e 10.406})$$

Put

$$\Lambda(f) = \text{ad } w_1 \circ \text{ad } w \circ \left( \sum_{j=1}^L \bar{\lambda}_j(f) \right) \quad \text{and} \quad \phi_2(f) = \text{ad } w_1 \circ \text{ad } w \circ \left( \sum_{j=1}^L \lambda_j(f) \right) \quad (\text{e 10.407})$$

for all  $f \in C(X)$ . Note that there exists a finite dimensional  $C^*$ -algebra  $C_0 (\cong \oplus_{j=1}^L C_j)$  such that  $\phi_2 : C(X) \rightarrow C_0$  unittally. We have

$$\|\phi(f) - [E(\phi(f))E + \phi_2(f) + \sum_{i=1}^m f(x_i)\bar{e}_i + \Lambda(f) + \phi'_1(f)]\| < \delta_1 \quad \text{and} \quad (\text{e 10.408})$$

$$\|\text{ad } u \circ \psi(f) - [E(\text{ad } u \circ \psi(f))E + \phi_2(f) + \sum_{i=1}^m f(x_i)\bar{e}_i + \Lambda(f) + \psi'_1(f)]\| < \delta_1 \quad (\text{e 10.409})$$

for all  $f \in \mathcal{G}_1$ , where  $\bar{e}_i = (e_i - e'_i - e''_i) + w_1^*(w^* e'_i w) w_1$ ,  $i = 1, 2, \dots, m$ .

Note that, by (e 10.399) and (e 10.393),

$$t_{j,x}(\pi_j(\bar{e}_i)) > \frac{3\Delta(\eta)}{8} - 2t_{j,x}(1_{C_j}) \quad (\text{e 10.410})$$

$$\geq \frac{3\Delta(\eta/2s)}{8} - \frac{2\Delta(\eta/2s)}{24N_1(N+2)(K+2)} > \frac{\Delta(\eta/2s)}{4} \quad (\text{e 10.411})$$

for all  $x \in X_j$  and  $j = 1, 2, \dots, L$ . Since

$$(N+1)(\tau(E) + t_{j,x}(1_{C_j})) + \sigma_2/16 < \quad (\text{e 10.412})$$

$$(N+1)\left(\frac{\sigma_2}{4} + \frac{\Delta(\eta_2/2s)}{24N_1(N+2)(K+2)}\right) + \sigma_2/16 < t_{j,x}(\pi_j(\bar{e}_i)), \quad (\text{e 10.413})$$

by (e 10.385), there is a projection  $p_{j,i} \leq \pi_j(\bar{e}_i)$  such that

$$(N+1)(\tau(E) + t_{j,x}(1_{C_j})) + \sigma_2/16 \geq t_{j,x}(p_{j,i}) > (N+1)(\tau(E) + t_{j,x}(1_{C_j})). \quad (\text{e 10.414})$$

Set  $p_i = \sum_{j=1}^L p_{j,i}$ .

We compute that

$$t_{j,x}(\pi_j(p_i)) = t_{j,x}(p_{j,i}) < (N+1)\sigma_2/4 + \frac{\Delta(\eta/2s)}{24N_1(K+2)} + \sigma_2/16 \quad (\text{e 10.415})$$

$$= \frac{35\Delta(\eta/2s)}{3 \cdot 4 \cdot 64 \cdot N_1(K+2)} + \frac{\sigma_2}{16} \quad (\text{e 10.416})$$

$$\text{for all } x \in X_j, j = 1, 2, \dots, L, \quad (\text{e 10.417})$$

$$\tau(p_i) > (N+1)(\tau(E) + \tau(1_{C_j})) \text{ and} \quad (\text{e 10.418})$$

$$t_{j,x}(1_{C_j}) + \sum_{i=1}^m t_{j,x}(\pi_j(p_i)) = t_{j,x}(1_{C_j}) + \sum_{i=1}^m t_{j,x}(\pi_j(p_i)) \quad (\text{e 10.419})$$

$$\leq \frac{\Delta(\eta_2/2s)}{24N_1(N+2)(K+2)} + \frac{m35\Delta(\eta/2s)}{3 \cdot 4 \cdot 64 \cdot N_1(K+2)} + \frac{m\sigma_2}{16} \quad (\text{e 10.420})$$

$$< \frac{\Delta(\eta_2/2s)}{3 \cdot 4 \cdot 64(K+2)} \quad (\text{e 10.421})$$

for all  $\tau \in T(A)$ . Since  $N_1 \geq 32m$ , from (e 10.411), (e 10.414) (and (e 10.412)), it follows that

$$t_{j,x}(\bar{e}_1 - p_i) > K(\tau(E) + t_{j,x}(1_{C_j}) + \sum_{i=1}^m t_{j,x}(p_i)) \quad (\text{e 10.422})$$

for all  $x \in X_j$ ,  $j = 1, 2, \dots, L$ . Define

$$B = (1_{B_1} - \sum_{i=1}^m p_i)B_1((1_{B_1} - \sum_{i=1}^m p_i). \quad (\text{e 10.423})$$

Define

$$\phi_1''(f) = \sum_{i=1}^m f(x_i)(\bar{e}_i - p_i) + \Lambda(f) + \phi_1'(f) \text{ and} \quad (\text{e 10.424})$$

$$\psi_1''(f) = \sum_{i=1}^m f(x_i)(\bar{e}_i - p_i) + \Lambda(f) + \psi_1'(f) \quad (\text{e 10.425})$$

for all  $f \in C(X)$ . So we view  $\phi_1''$  and  $\psi_1''$  as maps from  $C(X)$  into  $B$ . Note that, by (e 10.396),

$$[\phi_1'']|_{\mathcal{P}} = [\pi_\xi]|_{\mathcal{P}}, \quad (\text{e 10.426})$$

where  $\pi_\xi$  is the point-evaluation at  $\xi$ . By (e 10.388), we also have

$$[\psi_1'']|_{\mathcal{P}} = [\pi_\xi]|_{\mathcal{P}}. \quad (\text{e 10.427})$$

From (e 10.391) and (e 10.415), by the choices of  $\delta_1$  and  $\mathcal{G}_1$ , we have

$$\mu_{T \circ \phi_1''}(O_a) \geq \frac{63\Delta(a)}{64} - 79\sigma_2/48 \geq 2\Delta_1(a) \text{ and} \quad (\text{e 10.428})$$

$$\mu_{T \circ \psi_1''}(O_a) \geq 2\Delta_1(a) \text{ for all } T \in T(B) \quad (\text{e 10.429})$$

and for all  $a \geq \eta_1$ .

When  $X \in \mathbf{X}_0$ , by applying 8.5, we obtain unital homomorphisms  $\phi_1, \phi_2 : C(X) \rightarrow B$  such that

$$\|\phi_1(f) - \phi_1''(f)\| < \epsilon/16 \text{ and} \quad (\text{e 10.430})$$

$$\|\psi_1(f) - \psi_1''(f)\| < \epsilon/16 \quad (\text{e 10.431})$$

for all  $f \in \mathcal{F}_0$ . By the choice of  $\epsilon_2$  and  $\mathcal{F}_0$ , we have

$$\mu_{T \circ \phi_1}(O_a) \geq \Delta_1(a) \text{ and } \mu_{T \circ \psi_1}(O_a) \geq \Delta_1(a) \quad (\text{e 10.432})$$

for all  $a \geq \eta_0$ . Furthermore, we may also assume that

$$|T \circ \phi_1(f) - T \circ \psi_1(g)| < \epsilon \quad (\text{e 10.433})$$

for all  $f \in \mathcal{F}$  and  $T \in T(B)$ . Thus lemma follows by combining (e 10.430), (e 10.431) with (e 10.408), (e 10.409), (e 10.422) and (e 10.417), as well as (e 10.383).

When  $X \in \mathbf{X} \setminus \mathbf{X}_0$ , we will use (e 10.386) and (e 10.387). By considering each  $\phi_{1,0}'$  and  $\psi_{1,0}'$  individually (with a modification), by applying 8.4 and 8.6 in stead of 8.3, we also obtain  $\phi_1$  and  $\psi$  satisfying (e 10.430) and (e 10.431) as desired.  $\square$

**Remark 10.6.** When  $X = I \times \mathbb{T}$  or  $X$  is a connected one dimensional finite CW complex, the proof of 10.5 is much easier. In the case that  $X = I \times \mathbb{T}$ , a contractive completely positive linear map  $\phi : C(X) \rightarrow B$ , where  $B = \oplus_{j=1}^s C(X_j, M_{r(j)})$  with  $X_j = [0, 1]$  or a point, has the following property:

$$[\phi] = [\pi_\xi],$$

if  $\phi$  is unital  $\delta$ - $\mathcal{G}$ -multiplicative for some small  $\delta > 0$  and some finite subset  $\mathcal{G} \subset C(X)$ , where  $\pi_\xi(f) = f(\xi) \cdot 1_B$  for all  $f \in C(X)$  and  $\xi \in X$ . So 8.5 can be applied directly.

**Corollary 10.7.** *Let  $X \in \mathbf{X}$ . Let  $\epsilon > 0$ , let  $\eta_0 > 0$ , let  $\mathcal{F} \subset C(X)$  and let  $\Delta : (0, 1) \rightarrow (0, 1)$  be a non-decreasing map. Then there exists  $\eta > 0$ ,  $\delta > 0$ , a finite subset  $\mathcal{G} \subset C(X)$  satisfying the following:*

*Suppose that  $A$  is a unital separable simple  $C^*$ -algebra with  $TR(A) \leq 1$  and  $\phi : C(X) \rightarrow A$  is a unital  $\delta$ - $\mathcal{G}$ -multiplicative contractive completely positive linear map such that*

$$\mu_{\tau \circ \phi}(O_a) \geq \Delta(a) \text{ for all } a \geq \eta. \quad (\text{e 10.434})$$

*Then, for any  $\epsilon_0 > 0$ , for any integer  $K \geq 1$ , there are mutually orthogonal projections  $P_0$ ,  $P_1$  and  $P_2$  with  $P_0 + P_1 + P_2 = 1_A$ , there exists a unital  $C^*$ -subalgebra  $B = \oplus_{j=1}^s C(X_j, M_{r(j)})$*



with  $P_1 = 1_B$ , where  $X_j = [0, 1]$ , or  $X_j$  is a point, a finite dimensional  $C^*$ -subalgebra  $D$ , a unital completely positive linear map  $\phi_2 : C(A) \rightarrow D$  and there exists a unital homomorphism  $\phi_1 : C(X) \rightarrow B$  such that

$$\|\phi(f) - (P_0\phi(f)P_0 + \phi_2(f) + \phi_1(f))\| < \epsilon \text{ for all } f \in \mathcal{F} \quad (\text{e 10.435})$$

and

$$K\tau(P_0 + P_2) < \tau(P_1) \text{ for all } \tau \in T(A). \quad (\text{e 10.436})$$

Moreover, for any finite subset  $\mathcal{H} \subset A$ , one may require that

$$\|aP_0 - P_0a\| < \epsilon_0 \text{ for all } a \in \mathcal{H} \cup \phi(\mathcal{F}). \quad (\text{e 10.437})$$

*Proof.* Choose  $\psi = \phi$  and then apply 10.5. □

**Theorem 10.8.** *Let  $X$  be a finite simplicial complex in  $\mathbf{X}$ . Let  $\epsilon > 0$ , let  $\mathcal{F} \subset C(X)$  be a finite subset and let  $\Delta : (0, 1) \rightarrow (0, 1)$  be a non-decreasing map. There exists  $\eta > 0$ ,  $\delta > 0$ , a finite subset  $\mathcal{G} \subset C(X)$  a finite subset  $\mathcal{P} \subset \underline{K}(C(X))$  and a finite subset  $\mathcal{U} \subset \mathcal{U}(M_\infty(C(X)))$  satisfying the following:*

*Suppose that  $A$  is a unital separable simple  $C^*$ -algebra with tracial rank no more than one and  $\phi, \psi : C(X) \rightarrow A$  are two unital  $\delta$ - $\mathcal{G}$ -multiplicative contractive completely positive linear maps such that*

$$\mu_{\tau \circ \phi}(O_a) \geq \Delta(a) \text{ for all } a \geq \eta, \quad (\text{e 10.438})$$

$$|\tau \circ \phi(g) - \tau \circ \psi(g)| < \delta \text{ for all } g \in \mathcal{G}, \quad (\text{e 10.439})$$

*for all  $\tau \in T(A)$ ,*

$$[\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}} \text{ and } \text{dist}(\phi^\dagger(\bar{z}), \psi^\dagger(\bar{z})) < \delta \quad (\text{e 10.440})$$

*for all  $z \in \mathcal{U}$ . Then there exists a unitary  $u \in A$  such that*

$$\text{ad } u \circ \psi \approx_\epsilon \phi \text{ on } \mathcal{F}. \quad (\text{e 10.441})$$

*Proof.* Let  $\eta_1 > 0$  be as in 3.5 for  $\epsilon/4$  and  $\mathcal{F}$ . Let  $\sigma_1 = \Delta(\eta_1)/4\eta_1$ . Let  $\eta_0 > 0$  (in place of  $\eta$ ) and  $K_1 \geq 1$  (in place of  $K$ ) be as in 3.5 for  $\epsilon/4$  and  $\mathcal{F}$  above. Let  $\sigma_0 = \Delta(\eta_0)/4\eta_0$  (in place of  $\sigma$ ). Let  $\delta_1 > 0$  (in place of  $\delta$ ),  $\mathcal{G}_1 \subset C(X)$  (in place of  $\mathcal{G}$ ),  $\mathcal{P}_1 \subset \underline{K}(C(X))$  (in place of  $\mathcal{P}$ ),  $\mathcal{U}_1 \subset \mathcal{U}(M_\infty(C(X)))$  (in place of  $\mathcal{U}$ ) and  $L_1 \geq 1$  (in place of  $L$ ) be finite subsets required by 3.5.

Let  $L = 8\pi + 1$ . Let  $\delta_2 > 0$  (in place of  $\delta$ ),  $\mathcal{G}_2 \subset C(X)$  (in place of  $\mathcal{G}$ ),  $\mathcal{P}_2 \subset \underline{K}(C(X))$  (in place of  $\mathcal{P}$ ),  $\mathcal{U}_2 \subset \mathcal{U}(M_\infty(C(X)))$  (in place of  $\mathcal{U}$ ),  $l \geq 1$  and  $\epsilon_1 > 0$  be as required by 3.4 for  $\epsilon/4$  and  $\mathcal{F}$ . Let  $\epsilon_2 = \min\{\delta_1/2, \delta_2/2\}$  and  $\mathcal{F}_2 = \mathcal{F} \cup \mathcal{G}_1 \cup \mathcal{G}_2$ . Let  $\epsilon_3 > 0$  be a number smaller than  $\epsilon_2$ . Let  $N = l$  and  $K > 16/\min\{\sigma\eta, \sigma_1\eta_1, \delta_1\}$ . Let  $\eta_2 > 0$ , let  $\delta_3 > 0$  (in place of  $\delta$ ), let  $\mathcal{G}_3 \subset C(X)$  (in place of  $\mathcal{G}$ ), let  $\mathcal{P}_3 \subset \underline{K}(C(X))$  be required by 10.5 for  $\epsilon_3$  (in place of  $\epsilon$ ),  $\epsilon_1$ ,  $\min\{\eta_1, \eta_0\}$  (in place of  $\eta_0$ ) and  $\mathcal{F}_2$  (in place of  $\mathcal{F}$ ).

Let  $\eta = \min\{\eta_1, \eta_0, \eta_2\}$  and let  $\delta_4 = \min\{\Delta(\eta)/4, \delta_3, 1/32K_1\pi\}$ .

Let  $\delta$  be a positive number which is smaller than  $\delta_4$  and let  $\mathcal{G}$  be a finite subset containing  $\mathcal{G}_3$ . Let  $\mathcal{P} \subset \underline{K}(C(X))$  be a finite which contains  $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$  and the image of  $\mathcal{U}$  in  $\underline{K}(C(X))$ .

Suppose that  $A$  is a unital separable simple  $C^*$ -algebra with tracial rank one or zero and suppose  $\phi, \psi : C(X) \rightarrow A$  are two unital  $\delta$ - $\mathcal{G}$ -multiplicative contractive completely positive linear maps which satisfy the assumption of the theorem for the above  $\delta$ ,  $\mathcal{G}$ ,  $\mathcal{P}$  and  $\mathcal{U}$ .

It follows from 10.5 that there are four mutually orthogonal projections  $P_0, P_1, P_2$  and  $P_3$  with  $P_0 + P_1 + P_2 + P_3 = 1_A$ , there is a unital  $C^*$ -subalgebra  $B_1 \subset (P_1 + P_2 + P_3)A(P_1 + P_2 + P_3)$  with  $1_{B_1} = P_1 + P_2 + P_3$  and  $P_1, P_2, P_3 \in B_1$ , where  $B_1$  has the form  $B_1 = \oplus_{j=1}^s C(X_j, M_{r(j)})$  and where  $X_j = [0, 1]$ , or  $X_j$  is a point, there are unital homomorphisms  $\phi_1, \psi_1 : C(X) \rightarrow P_3 B_1 P_3$ , there exists a finite dimensional  $C^*$ -subalgebra  $C_0 \subset P_1 B_1 P_1$  with  $1_{C_0} = P_1$ , there exists a unital  $\epsilon_3$ - $\mathcal{F}_2$ -multiplicative contractive completely positive linear map  $\phi_2 : C(X) \rightarrow C_0$  and mutually orthogonal projections  $p_1, p_2, \dots, p_m \in B_1$  and a unitary  $v \in A$  such that

$$\|\phi(f) - [P_0 \phi(f) P_0 + \phi_2(f) + \sum_{i=1}^m f(x_i) p_i + \phi_1(f)]\| < \epsilon_3/2 \quad \text{and} \quad (\text{e } 10.442)$$

$$\|\text{ad } v \circ \psi(f) - [P_0 (\text{ad } v \circ \psi(f)) P_0 + \phi_2(f) + \sum_{i=1}^m f(x_i) p_i + \psi_1(f)]\| < \epsilon_3/2 \quad (\text{e } 10.443)$$

for all  $f \in \mathcal{F}_2$ , where  $\{x_1, x_2, \dots, x_m\}$  is  $\epsilon_1$ -dense in  $X$  and  $P_2 = \sum_{i=1}^m p_i$ ,

$$N\tau(P_0 + P_1) < \tau(p_i), \quad Kt_{j,x}(P_1 + P_2) \leq t_{j,x}(P_3) \quad (\text{e } 10.444)$$

$$\mu_{T \circ \phi_1}(O_a) \geq \Delta(a)/4, \quad \mu_{T \circ \psi_1}(O_a) \geq \Delta(a)/4 \quad \text{for all } a \geq \min\{\eta_0, \eta_1\} \quad (\text{e } 10.445)$$

$$\text{and } |T \circ \phi_1(f) - T \circ \psi_1(f)| < \epsilon_3 \quad \text{for all } f \in \mathcal{F}_2, \quad (\text{e } 10.446)$$

for all  $\tau \in T(A)$ ,  $x \in X_j$ ,  $j = 1, 2, \dots, m$  and for all  $T \in T(B)$ . Moreover, for any finite subset  $\mathcal{H} \subset A$ , one may require that

$$\|aP_0 - P_0a\| < \epsilon_3 \quad \text{and} \quad (1 - P_0)a(1 - P_0) \in_{\epsilon_3} B_1 \quad \text{for all } a \in \mathcal{H}. \quad (\text{e } 10.447)$$

We may also assume that  $r(j) \geq L_1$  for  $j = 1, 2, \dots, s$ . Put  $\phi_0(f) = P_0 \phi(f) P_0$ ,  $\psi_0(f) = P_0 (\text{ad } u \circ \psi(f)) P_0$ ,  $\phi_3(f) = \phi_2(f) + \sum_{i=1}^m f(x_i) p_i + \phi_1(f)$  and  $\psi_3(f) = \phi_2(f) + \sum_{i=1}^m f(x_i) p_i + \psi_1(f)$  for  $f \in C(X)$ .

Since

$$\text{dist}(\phi^\dagger(\bar{z}), \psi^\dagger(\bar{z})) < \delta \quad \text{for all } z \in \mathcal{U}, \quad (\text{e } 10.448)$$

with a sufficiently large  $\mathcal{H}$  (and sufficiently small  $\epsilon_3$ ), by 6.2 of [26], we may assume that

$$\text{dist}(\phi_0^\dagger(\bar{z}), \psi_0^\dagger(\bar{z})) < 2\delta \quad (\text{e } 10.449)$$

for all  $z \in \mathcal{U}$ . Furthermore, we may also assume that

$$\text{dist}(\phi_3^\dagger(\bar{z}), \psi_3^\dagger(\bar{z})) < 2\delta \quad (\text{e } 10.450)$$

for all  $z \in \mathcal{U}$ . Denote by  $D$  the determinant function on  $B_1$ . We compute that

$$D(\phi_1(z)\psi_1(z)^*) < 4\delta \quad \text{for all } z \in \mathcal{U}. \quad (\text{e } 10.451)$$

It follows that

$$\text{dist}(\phi_1^\dagger(\bar{z}), \psi_1^\dagger(\bar{z})) < 1/8K_1\pi \quad \text{for all } z \in \mathcal{U}. \quad (\text{e } 10.452)$$

We may also assume (with sufficiently large  $\mathcal{U}$  and sufficiently small  $\epsilon_3$ ) that

$$[\phi_1]_{\mathcal{P}} = [\psi_1]_{\mathcal{P}} \quad \text{and} \quad (\text{e } 10.453)$$

$$[\phi_0]_{\mathcal{P}} = [\psi_0]_{\mathcal{P}}. \quad (\text{e } 10.454)$$

By (e 10.445), (e 10.446) and (e 10.452) and by applying 3.5, we obtain a unitary  $w_1 \in B$  such that

$$\text{ad } w_1 \circ \psi_1 \approx_{\epsilon/4} \phi_1 \text{ on } \mathcal{F}. \quad (\text{e 10.455})$$

By applying 3.4, we also have a unitary  $w_2 \in (P_0 + P_2)A(P_0 + P_2)$  such that

$$\|w_2^*(\psi_0(f) \oplus \sum_{i=1}^m f(x_i)p_i)w_2 - (\phi_0(f) \oplus \sum_{j=1}^m f(x_j)p_j)\| < \epsilon/4 \text{ for all } f \in \mathcal{F}. \quad (\text{e 10.456})$$

The theorem then follows from the combination of (e 10.442), (e 10.443), (e 10.455) and (e 10.456).  $\square$

**Definition 10.9.** Let  $C = PM_k(C(X))P$  for some finite CW complex  $X$  and for some projection  $P \in M_k(C(X))$ . Suppose that the rank of  $P$  is  $m$ . Let  $t$  be a state on  $C$ . Then there is a Borel probability measure  $\mu_t$ , such that

$$t(f) = \int_X L_x(f(x))d\mu_t \text{ for all } f \in C, \quad (\text{e 10.457})$$

where  $L_x$  is a state on  $M_m$ . If  $t \in T(C)$ , then  $L_x(f(x)) = \text{tr}(f(x))$ , where  $\text{tr}$  is the normalized trace on  $M_m$ . There is an integer  $n \geq 1$  and a rank one trivial projection  $e \in M_n(C)$  such that  $eM_n(C)e \cong C(X)$ . It follows that there is a unitary  $u \in M_n(C)$  and a projection  $Q \in M_{kn}(C)$  such that  $u^*Cu = QM_k(eM_n(C)e)Q$ .

Suppose that  $A$  is a unital  $C^*$ -algebra,  $s$  is a state on  $A$  and suppose that  $\phi : C \rightarrow A$  is a contractive completely positive linear map. Then

$$s \circ \phi(f) = \int_X L_x(f(x))d\mu_{\tau \circ \phi} \text{ for all } f \in C,$$

where  $L_x$  is a state on  $M_m$ .

Let  $\tau \in T(C)$  and let  $\phi^{(n)} : M_n(C) \rightarrow M_n(A)$  be the homomorphism induced by  $\phi$ . Denote by  $\tilde{\phi} : C(X) \rightarrow \phi^{(n)}(e)M_n(A)\phi^{(n)}(e)$  the restriction of  $\phi^{(n)}$  on  $eM_n(C)e$ . It follows that the probability measure  $\mu_{\tau \circ \tilde{\phi}}$  induced by  $\tau \circ \tilde{\phi}$  is equal to  $\mu_{\tau \circ \phi}$ .

**Corollary 10.10.** Let  $X$  be a finite simplicial complex in  $\mathbf{X}$ . Let  $\epsilon > 0$ , let  $\mathcal{F} \subset C = PM_k(C(X))P$ , where  $P \in M_k(C(X))$  is a projection, be a finite subset and let  $\Delta : (0, 1) \rightarrow (0, 1)$  be a non-decreasing map. There exists  $\eta > 0$ ,  $\delta > 0$ , a finite subset  $\mathcal{G}$ , a finite subset  $\mathcal{P} \subset \underline{K}(C)$  and a finite subset  $\mathcal{U} \subset \mathcal{U}(M_\infty(C))$  satisfying the following:

Suppose that  $A$  is a unital separable simple  $C^*$ -algebra with tracial rank no more than one and  $\phi, \psi : C \rightarrow A$ , where are two unital  $\delta$ - $\mathcal{G}$ -multiplicative contractive completely positive linear maps such that

$$\mu_{\tau \circ \phi}(O_a) \geq \Delta(a) \text{ for all } a \geq \eta, \quad (\text{e 10.458})$$

$$|\tau \circ \phi(g) - \tau \circ \psi(g)| < \delta \text{ for all } g \in \mathcal{G}, \quad (\text{e 10.459})$$

for all  $\tau \in T(A)$ ,

$$[\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}} \text{ and } \text{dist}(\phi^\dagger(\bar{z}), \psi^\dagger(\bar{z})) < \delta \quad (\text{e 10.460})$$

for all  $z \in \mathcal{U}$ . Then there exists a unitary  $u \in A$  such that

$$\text{ad } u \circ \psi \approx_\epsilon \phi \text{ on } \mathcal{F}. \quad (\text{e 10.461})$$

*Proof.* It is standard (using 10.9) that the general case can be reduced to the case that  $C = M_k(C(X))$ . It is then clear that this corollary follows from 10.8.  $\square$

**10.11.** It should be noted in the case that  $X = I \times \mathbb{T}$ , or  $X$  is an  $n$ -dimensional torus, in the above 10.8 and 10.10, one may only consider  $\mathcal{U} \subset U(C)$ . Moreover, in the case that  $X$  is a finite simplicial complex with torsion  $K_1(C(X))$ , then the map  $\phi^\dagger$  and  $\psi^\dagger$  can be removed entirely (see Corollary 2.14 of [11]).

## 11 AH-algebras

Let  $X$  be a compact metric space and let  $A$  be a unital simple  $C^*$ -algebra with  $T(A) \neq \emptyset$ . Suppose that  $\phi : C(X) \rightarrow A$  is a unital monomorphism. Then  $\mu_{\tau \circ \phi}$  is a strictly positive probability Borel measure. Fix  $a \in (0, 1)$ . Let  $\{x_1, x_2, \dots, x_m\} \subset X$  be an  $a/4$ -dense subset. Define

$$d(a, i) = (1/2) \inf \{ \mu_{\tau \circ \phi}(B_{a/4}(x_i)) : \tau \in T(A) \}, \quad i = 1, 2, \dots, m.$$

Fix a non-zero positive function  $g \in C(X)$  with  $g \leq 1$  whose support contained in  $B_{a/4}(x_i)$ . Then, since  $A$  is simple,  $\inf \{ \tau(\phi(g)) : \tau \in T(A) \} > 0$ . It follows that  $d(a, i) > 0$ . Put

$$\Delta(a) = \min \{ d(a, i) : i = 1, 2, \dots, m \}.$$

For any  $x \in X$ , there exists  $i$  such that  $B_a(x) \supset B_{a/4}(x_i)$ . Thus

$$\mu_{\tau \circ \phi}(B_a(x)) \geq \Delta(a) \text{ for all } \tau \in T(A). \quad (\text{e 11.462})$$

Note that  $\Delta$  gives a non-decreasing map from  $(0, 1) \rightarrow (0, 1)$ .

This proves the following:

**Proposition 11.1.** *Let  $X$  be a compact metric space and let  $A$  be a unital simple  $C^*$ -algebra with  $T(A) \neq \emptyset$ . Suppose that  $\phi : C(X) \rightarrow A$  is a unital monomorphism. Then there is a non-decreasing map  $\Delta : (0, 1) \rightarrow (0, 1)$  such that*

$$\mu_{\tau \circ \phi}(O_a) \geq \Delta(a) \text{ for all } \tau \in T(A) \quad (\text{e 11.463})$$

for all open balls  $O_a$  of  $X$  with radius  $a \in (0, 1)$ .

**Definition 11.2.** Let  $C$  be a  $C^*$ -algebra. Let  $T = N \times K : C_+ \setminus \{0\} \rightarrow \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$  be a map. Suppose that  $A$  is a unital  $C^*$ -algebra and  $\phi : C \rightarrow A$  is a homomorphism. Let  $\mathcal{H} \subset C_+ \setminus \{0\}$  be a finite subset. We say that  $\phi$  is  $T$ - $\mathcal{H}$ -full if there are  $x_{a,i} \in A$ ,  $i = 1, 2, \dots, N(a)$  with  $\|x_{a,i}\| \leq K(a)$ ,  $i = 1, 2, \dots, N(a)$ , such that

$$\sum_{i=1}^{N(a)} x_{a,i}^* \phi(a) x_{a,i} = 1_A$$

for all  $a \in \mathcal{H}$ . The homomorphism  $\phi$  is said to be  $T$ -full, if

$$\sum_{i=1}^{N(a)} x_{a,i}^* \phi(a) x_{a,i} = 1_A$$

for all  $a \in A_+ \setminus \{0\}$ . If  $\phi$  is  $T$ -full, then  $\phi$  is injective.

**Proposition 11.3.** *Let  $X$  be a finite CW complex, let  $P \in M_k(C(X))$  be a projection and let  $C_1 = PM_k(C(X))P$ . Suppose that  $T = N \times N : C_+ \setminus \{0\} \rightarrow \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$  is a map. Then there exists a non-decreasing map  $\Delta : (0, 1) \rightarrow (0, 1)$  associated with  $T$  satisfying the following:*

*For any  $\eta > 0$ , there is a finite subset  $\mathcal{H} \subset (C_1 \otimes C(\mathbb{T}))_+ \setminus \{0\}$  such that, for any unital  $C^*$ -algebra  $B$  with  $T(B) \neq \emptyset$  and any unital contractive completely positive linear map  $\phi : C \rightarrow B$  which is  $T$ - $\mathcal{H}$ -full, one has that*

$$\mu_{\tau \circ \phi}(O_a) \geq \Delta(a) \text{ for all } a \geq \eta \text{ for all } a \in (\eta, 1). \quad (\text{e 11.464})$$

*Proof.* To simplify notation, using 10.9, without loss of generality, we may assume that  $C = C(X)$ . Fix  $1 > a > 0$ . Let  $\{x_1, x_2, \dots, x_n\}$  be an  $a/4$ -dense subset of  $X$ . Let  $f_i$  be a positive function in  $C(X)$  with  $0 \leq f_i \leq 1$  whose support is in  $B_{a/4}(x_i)$  and contains  $B_{a/6}(x_i)$ ,  $i = 1, 2, \dots, m$ . Define  $\Delta' : (0, 1) \rightarrow (0, 1)$  by

$$\Delta'(a) = \frac{1}{\max\{N(f_i)K(f_i)^2 : 1 \leq i \leq m\}}. \quad (\text{e 11.465})$$

$$(\text{e 11.466})$$

Define

$$\Delta(a) = \min\{\Delta'(b) : b \geq a\}.$$

It is clear that  $\Delta$  is non-decreasing.

Now, let  $B$  be a unital  $C^*$ -algebra with  $T(B) \neq \emptyset$  and let  $\phi : C \rightarrow B$  be a unital contractive completely positive linear map which is  $T$ - $\mathcal{H}$ -full. For each  $i$ , there are  $x_{i,j}$ ,  $j = 1, 2, \dots, N(f_i)$ , with  $\|x_{i,j}\| \leq N(f_i)$  such that

$$\sum_{j=1}^{N(f_i)} x_{i,j}^* \phi(f_i) x_{i,j} = 1_B, \quad i = 1, 2, \dots, m. \quad (\text{e 11.467})$$

Fix a  $\tau \in T(B)$ . There exists  $j$  such that

$$\tau(x_{i,j}^* \phi(f_i) x_{i,j}) \geq \frac{1}{N(f_i)}. \quad (\text{e 11.468})$$

It follows that

$$\|x_{i,j} x_{i,j}^*\| \tau(\phi(f_i)) \geq \tau(\phi(f_i)^{1/2} x_{i,j} x_{i,j}^* \phi(f_i)^{1/2}) \quad (\text{e 11.469})$$

$$= \tau(x_{i,j}^* \phi(f_i) x_{i,j}) \geq \frac{1}{N(f_i)}. \quad (\text{e 11.470})$$

It follows that

$$\tau(\phi(f_i)) \geq \frac{1}{N(f_i)K(f_i)^2}. \quad (\text{e 11.471})$$

This holds for all  $\tau \in T(B)$ ,  $i = 1, 2, \dots, m$ . Now for any open ball  $O_a$  with radius  $a$ . Suppose that  $y$  is the center. Then  $y \in B_{a/4}(x_i)$  for some  $1 \leq i \leq m$ . Thus

$$O_a \supset B_{a/4}(x_i).$$

It follows that

$$\mu_{\tau \circ \phi}(O_a) \geq \tau(f_i) \geq \frac{1}{N(f_i)K(f_i)^2} \geq \Delta(a) \quad (\text{e 11.472})$$

for all  $\tau \in T(B)$ . It is then clear that, when  $\eta > 0$  is given, such finite subset  $\mathcal{H}$  exists.  $\square$

**Definition 11.4.** An AH-algebra  $C$  is said to have property (J) if  $C$  is isomorphic to an inductive limit  $\lim_{n \rightarrow \infty} (C_n, \phi_j)$ , where  $\bigoplus_{j=1}^{R(i)} P_{n,j} M_{r(n,j)}(C(X_{n,j})) P_{n,j}$ , where  $X_{n,j}$  is an one dimensional finite CW complex or a simplicial complex in  $\mathbf{X}$  and where  $P_{n,j} \in M_{r(n,j)}(C(X_{n,j}))$  is a projection, and each  $\phi_j$  is injective.

**Theorem 11.5.** Let  $C$  be a unital AH-algebra with property (J). Let  $T = N \times K : C_+ \setminus \{0\} \rightarrow \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$ . Then, for any  $\epsilon > 0$  and any finite subset  $\mathcal{F} \subset C$ , there exists  $\delta > 0$  and a finite subset  $\mathcal{G} \subset C$ , a finite subset  $\mathcal{P} \subset \underline{K}(C)$  and a finite subset  $\mathcal{U} \subset U(M_\infty(C))$  satisfying the following: Suppose that  $A$  is a unital separable simple  $C^*$ -algebra with tracial rank one or zero and  $\phi, \psi : C \rightarrow A$  are two unital  $\delta$ - $\mathcal{G}$ -multiplicative contractive completely positive linear maps such that  $\phi$  is  $T$ - $\mathcal{H}$ -full, where  $\mathcal{H} = \mathcal{G} \cap (C_+ \setminus \{0\})$ ,

$$|\tau \circ \phi(g) - \tau \circ \psi(g)| < \delta \text{ for all } g \in \mathcal{G} \quad (\text{e 11.473})$$

for all  $\tau \in T(A)$ ,

$$[\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}} \text{ and} \quad (\text{e 11.474})$$

$$\text{dist}(\phi^\dagger(\bar{z}), \psi^\dagger(\bar{z})) < \delta \quad (\text{e 11.475})$$

for all  $z \in \mathcal{U}$ . Then there exists a unitary  $u \in A$  such that

$$\text{ad } u \circ \psi \approx_\epsilon \phi \text{ on } \mathcal{F}. \quad (\text{e 11.476})$$

*Proof.* We may write  $C = \overline{\bigcup_{n=1}^\infty C_n}$ , where each  $C_n = \bigoplus_{j=1}^{m(n)} C_{n,j}$ ,  $C_{n,j} = P_{n,j} M_{r(n,j)}(C(X_{n,j})) P_{n,j}$ ,  $X_{n,j}$  is a point, a connected finite CW complex of dimension 1, or  $X_{n,j}$  is a finite simplicial complex in  $\mathbf{X}$  and  $P_{n,j} \in M_{r(n,j)}(C(X_{n,j}))$  is a projection.

Fix a finite subset  $\mathcal{F} \subset C$  and  $\epsilon > 0$ . Without loss of generality, we may assume that  $\mathcal{F} \subset C_n$  for some  $n \geq 1$ . Let  $p_1, p_2, \dots, p_{m(n)}$  be the identities of the each summand of  $C_n$ . Since  $A$  is stable rank one, conjugating a unitary, without loss of generality, we may assume that  $\phi(p_i) = \psi(p_i)$ ,  $i = 1, 2, \dots, m(n)$ . It is then clear that we may reduce the general case to the case that  $C_n$  has only one summand. Then the theorem follows from the combination of 10.10 and 11.3.  $\square$

**Corollary 11.6.** Let  $C$  be a unital AH-algebra with property (J) and let  $A$  be a unital simple  $C^*$ -algebra with  $TR(A) \leq 1$ . Suppose that  $\phi : C \rightarrow A$  is unital monomorphism. Then, for any  $\epsilon > 0$ , and finite subset  $\mathcal{F} \subset C$ , there exists  $\delta > 0$ , a finite subset  $\mathcal{P} \subset \underline{K}(C)$ , a finite subset  $\mathcal{U} \subset U(M_\infty(C))$  and a finite subset  $\mathcal{H} \subset C$  satisfying the following:

if  $\psi : C \rightarrow A$  is another unital monomorphism with

$$[\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}} \quad (\text{e 11.477})$$

$$\text{dist}(\phi^\dagger(\bar{z}), \psi^\dagger(\bar{z})) < \delta \text{ for all } z \in \mathcal{U} \text{ and} \quad (\text{e 11.478})$$

$$|\tau \circ \phi(g) - \tau \circ \psi(g)| < \delta \text{ for all } g \in \mathcal{H}, \quad (\text{e 11.479})$$

then there exists a unitary  $u \in A$  such that

$$\text{ad } u \circ \psi \approx_\epsilon \phi \text{ on } \mathcal{F}.$$

*Proof.* Write  $C = \overline{\bigcup_n C_n}$ , where each  $C_n$  is a finite direct sum of  $C^*$ -algebras with the form as described in 10.10. Fix a finite subset  $\mathcal{F}$  and  $\epsilon > 0$ . Without loss of generality, we may assume that  $\mathcal{F} \subset C_n$ . To simplify notation further, we may assume that  $C_n$  has the form  $PM_k(C(X))P$  for some  $X \in \mathbf{X}$ . Since  $\phi$  is a given monomorphism, by 11.1, there exists a non-decreasing map  $\Delta : (0, 1) \rightarrow (0, 1)$  such that

$$\mu_{\tau \circ \phi}(O_a) \geq \Delta(a)$$

for all  $a \in (0, 1)$ . Thus conclusion follows by applying 10.10.  $\square$

**Corollary 11.7.** *Let  $C$  be a unital AH-algebra with property (J) and let  $A$  be a unital simple  $C^*$ -algebra with  $TR(A) \leq 1$ . Suppose that  $\phi, \psi : C \rightarrow A$  are two unital monomorphisms. Then  $\phi$  and  $\psi$  are approximately unitarily equivalent if and only if*

$$[\phi] = [\psi] \text{ in } KL(C, A), \quad (\text{e 11.480})$$

$$\phi_{\#} = \psi_{\#} \text{ and } \phi^{\dagger} = \psi^{\dagger}. \quad (\text{e 11.481})$$

**Corollary 11.8.** *Let  $C$  be a unital separable simple  $C^*$ -algebra with  $TR(C) \leq 1$  and satisfying the UCT and let  $A$  be a unital simple  $C^*$ -algebra with  $TR(A) \leq 1$ . Suppose that  $\phi, \psi : C \rightarrow A$  are two unital homomorphisms. Then  $\phi$  and  $\psi$  are approximately unitarily equivalent if and only if*

$$[\phi] = [\psi] \text{ in } KL(C, A) \quad (\text{e 11.482})$$

$$\phi_{\#} = \psi_{\#} \text{ and } \phi^{\dagger} = \psi^{\dagger}. \quad (\text{e 11.483})$$

*Proof.* It follows from 10.9 of [26] (by applying a theorem of Villadsen [47]) that there exists a unital simple AH-algebra  $B$  with property (J) such that

$$(K_0(C), K_0(C)_+, [1_C], K_1(C), T(C)) \cong (K_0(B), K_0(B)_+, [1_B], K_1(B), T(B))$$

(see 10.2 of [26] for the meaning of the above). It follows from 10.4 of [26] that  $C \cong B$ . Thus the corollary follows.  $\square$

**Remark 11.9.** In 11.8,  $C$  is assumed to have tracial rank no more than one, in particular, it has stable rank one. Therefore, the maps  $\phi^{\dagger}$  and  $\psi^{\dagger}$  can be regarded as maps from  $U((C)/CU(C))$  to  $U(A)/CU(A)$  (no need to go to matrix algebras).

It should be noted that Corollary 11.8 can also be derived from results in [26]. It is important that in 11.5 and 11.6  $C^*$ -algebra  $C$  is not assumed to be simple, in particular,  $C$  could be commutative. These results will be used in subsequent papers where we study the so-called the Basic Homotopy Lemma ([34]) and asymptotic unitary equivalence ([35]) in simple  $C^*$ -algebras with tracial rank one.

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